Math 371 - Introductory Real Analysis

Sep 11 2019

0.3.4: Equivalence Classes

A Word about $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$

- We assume the existence of $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ with their standard order and algebraic properties.

- For elements $a/b, c/d \in \mathbb{Q} = \left\{ \frac{p}{q} \ : \ p, q \in \mathbb{Z}, q \neq 0 \right\}$,

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

- In this way, $\mathbb{Q}$ is a collection of equivalence classes.

Relations

- **Definition:** For a set A, a binary relation on A is a subset $\mathcal{R} \subset A \times A$.

- For example, consider $A = \mathbb{N}$ and the relation of being "less than". Here $\mathcal{R} = \{(a, b) \in \mathbb{N} \times \mathbb{N} \; : \; a < b\}$.

- **Definition:** A relation $\mathcal{R}$ on a set A is said to be an equivalence relation if it is
  (i) reflexive: $(a, a) \in \mathcal{R}$ for every $a \in A$,
  (ii) symmetric: $(a, b) \in \mathcal{R}$ if and only if $(b, a) \in \mathcal{R}$, and
  (iii) transitive: if $(a, b), (b, c) \in \mathcal{R}$ then $(a, c) \in \mathcal{R}$.

- For example, $\mathcal{R}_1 = \{(a, b) \in \mathbb{N} \times \mathbb{N} \; : \; a = b\}$ is an equivalence relation, but $\mathcal{R}_2 = \{(a, b) \in \mathbb{N} \times \mathbb{N} \; : \; a < b\}$ is not (why?).

Equivalence Classes

- **Definition:** For a set A on which an equivalence relation $\mathcal{R}$ is defined, we define the equivalence class containing a to be
$$[a] = \{x \in A \ : \ (a, x) \in \mathcal{R}\},$$
the set of elements of A that are equivalent to a under $\mathcal{R}$. Here a is a representative of the class.

- An equivalence relation $\mathcal{R}$ on a set A partitions A into disjoint equivalence classes.

Back to $\mathbb{Q}$

- $\mathbb{Q}$ can thus be defined as

  $$\mathbb{Q} = \{[(a, b)] \;:\; (a, b) \in \mathbb{Z} \times \mathbb{N}\}$$

  where the equivalence classes are given by the equivalence relation $((a, b), (c, d)) \in \mathcal{R} \iff ad = bc$.

- We normally just write a/b instead of [(a,b)] with the understanding that a/b represents all of the elements of $\mathbb{Q}$ equal to a/b.

- For example, "one half" may be represented by $1/2, 2/4, 3/6, \ldots$, but these are all representatives of a single element of $\mathbb{Q}$.

0.3.5: Cardinality

Idea

The cardinality of a set is one way to describe the size of the set. $\emptyset$ has zero elements, $\{\sqrt{2}, e, \pi\}$ has 3 elements– easy enough. But $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, etc, have infinitely many elements, yet the "sizes" of these infinities differ in some sense. We wish to make this notion precise.

Cardinality

Definition: We say that sets A and B have the same cardinality if there exists a bijection $f : A \to B$, in which case we write $|A| = |B|$.

1. If $A = \emptyset$, $|A| := 0$. If $|A| = |\{1, 2, \dots, n\}|$, $|A| := n$. In either case A is said to be finite.
2. $|A| \leq |B|$ if there is an injection from A to B.
3. $|A| < |B|$ if there is an injection from A to B but no surjection.
4. A is countably infinite or denumerable if $|A| = |\mathbb{N}|$.
5. A is countable if $|A| = |\mathbb{N}|$ or $|A| = n$.
6. A is uncountable if not countable.
7. $|\mathbb{N}| = \aleph_0$, called aleph-naught.

Theorems

- **Theorem (Cantor-Bernstein-Schröder):** $|A| = |B|$ if and only if $|A| \leq |B|$ and $|B| \leq |A|$.

- **Theorem:** If $A \subset B$ then $|A| \leq |B|$.