Math 371 - Introductory Real Analysis

Sep 9 2019

0.3.3: Functions

The Natural Numbers $\mathbb{N}$

- As a starting point for our work we assume the existence of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ together with their natural ordering

$$1 < 2 < 3 < 4 < \cdots$$

- We will assume the well Ordering Property of $\mathbb{N}$: Every nonempty subset of $\mathbb{N}$ has a smallest element.

- That is, for each nonempty $A \subset \mathbb{N}$, there is $x \in A$ such that $x \leq y$ for every $y \in A$.

Induction

- **Theorem:** Suppose $P(n)$ is a statement depending on a natural number n. If
  (i) $P(1)$ is true, and
  (ii) $P(n+1)$ is true whenever $P(n)$ is true

  then $P(n)$ is true for every $n \in \mathbb{N}$.

- Assumed knowledge for this course. See Section 0.3.2.

Functions

- **Definition:** For sets A and B, the Cartesian Product $A \times B$ is
$$A \times B = \{(x, y) \ : \ x \in A \text{ and } y \in B\}$$

- **Definition:** For sets A and B, a function $f : A \to B$ is a subset $f \subset A \times B$ such that for each $x \in A$ there is a unique (i.e. one and only one) $(x, y) \in f$, in which case we write $f(x) = y$.

- Often a function f is given by a formula, but this need not be the case.

- A functions is sometimes called a mapping

Domain and Range

Definition: Let $f : A \to B$ be a function.

- The set A is called the domain of f, written $\mathcal{D}(f)$.
- The range of f is the set
  $\mathcal{R}(f) = \{y \in B \ : \ f(x) = y \text{ for some } x \in A\}$.
- Notice $\mathcal{R}(f) \subset B$, and possibly $\mathcal{R}(f) \subsetneq B$.

Direct and Inverse Images

Definition: Let $f : A \to B$ be a function.

- For $C \subset A$ the direct image of C is $f(C) = \{f(x) \in B \ : \ x \in C\}$.
- For $D \subset B$ the inverse image of D is $f^{-1}(D) = \{x \in A \ : \ f(x) \in D\}$.
- Caution: writing $f^{-1}(D)$ is not meant to imply that f has an associated inverse function– more on inverse functions later.

Injective, Surjective, Bijective

Definition: Let $f : A \to B$ be a function.

- f is said to be injective or one-to-one or an injection if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- f is said to be surjective or onto or a surjection if $f(A) = B$. That is, for each $y \in B$ there is some $x \in A$ such that $f(x) = y$.
- If f is both injective and surjective it is said to be bijective or a bijection .

Inverse Functions

Definition: Let $f : A \to B$ be a bijective function. We define the function $f^{-1} : B \to A$ as follows: For $y \in B$, $f^{-1}(y) = x \in A$ such that $f(x) = y$.

f^{-1} defined in this way is indeed a function since

- If $y \in B$, $f^{-1}(\{y\})$ is nonempty by the surjectivity of f.
- If $x_1, x_2 \in f^{-1}(\{y\})$ then $f(x_1) = f(x_2) = y$, so that $x_1 = x_2$ since f is injective. That is, $f^{-1}(\{y\})$ contains one and only one element, which we have defined to be $f^{-1}(y)$.

Function Composition

Definition: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. We define the composition of g with f as the function $g \circ f : A \rightarrow C$ with property

$$(g \circ f)(x) = g(f(x))$$