Math 371 - Introductory Real Analysis

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What is Real Analysis?

- **Wikipedia:** Real analysis. . . has its beginnings in the rigorous formulation of calculus. It is a branch of mathematical analysis dealing with the set of real numbers. In particular, it deals with the analytic properties of real functions and sequences, including convergence and limits of sequences of real numbers, the calculus of the real numbers, and continuity, smoothness and related properties of real-valued functions.

- **mathematical analysis:** the branch of pure mathematics most explicitly concerned with the notion of a limit, whether the limit of a sequence or the limit of a function. It also includes the theories of differentiation, integration and measure, infinite series, and analytic functions.

In other words. . .

- In calculus, we learn how to apply tools (theorems) to solve problems (optimization, related rates, linear approximation) without worrying so much about the rigorous development of the tools themselves.

- In real analysis, we very carefully develop all the definitions and methods needed to prove these theorems and rigorously demonstrate the validity of the tools.

Thinking back to calculus...

- Most every important concept was defined in terms of limits: continuity, the derivative, the definite integral

- But, the notion of the limit itself was rather vague

- For example,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

  means

  $\sin(x)/x$ gets close to 1 as x gets close to 0. ☹

The key notion

- The key and subtle concept that makes calculus work is that of the limit

- Notion of a limit was truly a major advance in mathematics. Instead of thinking of numbers as only those quantities that could be calculated in a finite number of steps, a number could be viewed as the result of a process, a target that is "reachable" after an infinite number of steps.

What makes analysis different

- In the words of the author: “In algebra, we prove equalities directly. That is, we prove that an object (a number perhaps) is equal to another object. In analysis, we generally prove inequalities.”

- To illustrate: Suppose x is a real number.

  If $0 \leq x < \epsilon$ for every real number $\epsilon > 0$, then $x = 0$.

  That is, to show that a positive number is zero, it is enough to show that it is less than any other positive real number.

Example of a major result using analysis

Theorem (Fourier): Suppose f is a continuous function defined on the real numbers such that $f(x + 2\pi) = f(x)$ for every x, and suppose that f' is also continuous. Then

$$\begin{aligned} f(x) &= \frac{a_0}{2} + [a_1 \cos(x) + b_1 \sin(x)] + [a_2 \cos(2x) + b_2 \sin(2x)] + \\ &\quad [a_3 \cos(3x) + b_3 \sin(3x)] + \cdots \end{aligned}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

The plan

- **Proofs:** we want to prove things in analysis, so we'll start with a few words about proof techniques.

- **Set theory:** before we work with sets of real numbers we have to get comfortable with some basic set theory. This will be review for many, but we'll also introduce a few new concepts and see some neat results.

- **Real numbers:** the stars of the show. Limits and all that comes after depend on the structure and properties of real numbers.

The plan, continued

- **Sequences and Series:** properties of lists and sums of real numbers: our first real look at limits.
- **Continuous functions:** important properties of continuous functions which follow from their examination using limits.
- **The Derivative:** We now have the machinery to prove some of the major results: chain rule, mean value theorem, Taylor's theorem.

The plan, continued

- **The Riemann Integral:** Again, defined in terms of limits. Fundamental Theorem of Calculus.

- **Sequences of Functions:** We can extend our study of limits of sequences of real numbers to limits of sequences of functions. The theory is fundamental to many fields: differential equations, harmonic analysis, functional analysis, etc.

Writing Proofs

Advice from Dr. Francis Su, Harvey Mudd College

Communicating mathematics well is an important part of doing mathematics. As you write up your homework solutions, keep these things in mind:

- Write in complete sentences.
  Complete thoughts are sentences that end in periods. You may still highlight important equations by displaying them, but even displayed equations should have punctuation! Use paragraphs to separate important ideas.

- Use helpful connective phrases.
  "If", "then", "so", "therefore", "we see that", "recall that", ...
- Your audience is other students in the class who have not seen this problem before.
  Remind the reader of any relevant facts from class or the book. Your solution should give adequate detail so that the reader can follow your solution.

Advice from Dr. Francis Su, Harvey Mudd College

- It is possible to write too much!
  If you write out every triviality, the reader may get lost in the details. This is not good writing, either. (In particular, really trivial calculations need not be shown.)

- Avoid shorthand.
  Don't use arrows, and write out 'for all', 'there exists'.

- You may wish to outline your problem-solving strategy at the beginning of the problem.

Advice from Glen

- Structure your proofs by stating the full “Proposition” followed by “Proof:” as in class. Start each new proof on a new page. Indent your argument where appropriate for readability and be careful with notation.

- As with any proof, clarity of presentation is as important as solving the problem. Strive to make your proofs clear, concise and precise. Feel free to use results from earlier problems (or class) in subsequent ones.

0.3: Set Theory

Sets

- **Definition:** A set S is a collection of objects called members or elements.
- Write $x \in S$ if x is an element of S, $x \notin S$ if x is not an element of S.
- $S = \emptyset$ if S contains no elements.

Some Important Sets

- Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$
- Integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- Rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \ : \ p, q \in \mathbb{Z}, q \neq 0 \right\}$
- Real numbers: $\mathbb{R}$. More on the definition of $\mathbb{R}$ later.

Well Ordering Property of $\mathbb{N}$

We will assume the well Ordering Property of $\mathbb{N}$:

Every non-empty subset of $\mathbb{N}$ has a smallest element.