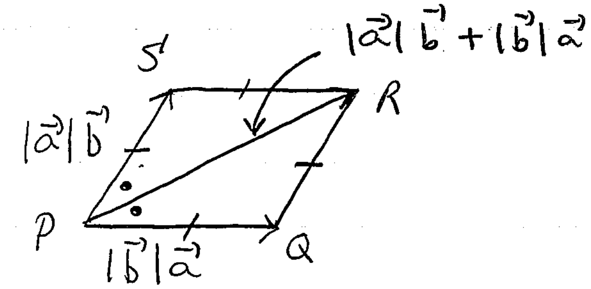
Review Problem 2: Sol^{n}s.

(1) $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is the diagonal of the parallelogram defined by the vectors $|\vec{a}|\vec{b}$ and $|\vec{b}|\vec{a}$.
Since $\left||\vec{a}|\vec{b}\right| = |\vec{a}||\vec{b}| = \left||\vec{b}|\vec{a}\right|$, all four sides of the parallelogram are equal:



The two resulting congruent triangles are isosceles, so $\angle_{SPR} = \angle_{RPQ}$. That is, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $|\vec{a}|\vec{b}$ and $|\vec{b}|\vec{a}$. Since $|\vec{a}|\vec{b}$ & $|\vec{b}|\vec{a}$ are scalar multiples of $\vec{b}$ & $\vec{a}$, respectively, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a}$ & $\vec{b}$.

(2) $(|\vec{b}|\vec{a} + |\vec{a}|\vec{b}) \cdot (|\vec{b}|\vec{a} - |\vec{a}|\vec{b})$

$= |\vec{b}|^2(\vec{a}\cdot\vec{a}) + \cancel{|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})} - \cancel{|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})} - |\vec{a}|^2(\vec{b}\cdot\vec{b})$

$= |\vec{b}|^2|\vec{a}|^2 - |\vec{a}|^2|\vec{b}|^2$

$= 0$

$\therefore$ $|\vec{b}|\vec{a} + |\vec{a}|\vec{b}$ and $\vec{b}|\vec{a}| + |\vec{a}|\vec{b}$ are orthogonal.

(3) $P(3,-1,2)$; $\vec{r}(t) = \langle 2,-1,0\rangle + t\langle 2,3,0\rangle$

$\vec{r}(0) = \langle 2,-1,0\rangle$ gives point $Q(2,-1,0)$ on plane ;
$\vec{r}(1) = \langle 4,2,0\rangle$ gives point $R(4,2,0)$ on plane.

Normal to plane is $\vec{n} = \overrightarrow{QP} \times \overrightarrow{RP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ -1 & -3 & 2 \end{vmatrix} = \langle 6,-4,-3\rangle$

$\therefore$ Using $P(3,-1,2)$ and $\vec{n} = \langle 6,-4,-3\rangle$,

$\langle x-3, y+1, z-2\rangle \cdot \langle 6,-4,-3\rangle = 0$

$\Rightarrow 6(x-3) - 4(y+1) - 3(z-2) = 0 \rightarrow 6x - 4y - 3z = 16$

Review Problem 2 : $sol^{\underline{n}}s$

(4) $8x+y+z=1$ has normal $\vec{n}_1 = \langle 8,1,1\rangle$.
$x-y-z=0$ has normal $\vec{n}_2 = \langle 1,-1,-1\rangle$.
$\vec{w} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \langle 0,9,-9\rangle$ is $\parallel$ to both planes.

$\therefore \vec{u} = \dfrac{\vec{w}}{|\vec{w}|} = \dfrac{\langle 0,9,-9\rangle}{9\sqrt{2}} = \langle 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\rangle$ is the required unit vector.

(5) Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ be the required vector.
Then $u_1^2+u_2^2+u_3^2=1$ and $\vec{u}\cdot\hat{i} = \cancel{|\vec{u}|}\cancel{|\hat{i}|}\cos(30^\circ) = \frac{\sqrt{3}}{2} = u_1$.
Letting α be the common angle between $\vec{u}, \hat{j}$ and $\vec{u}, \hat{k}$,
$\vec{u}\cdot\hat{j} = \cos(\alpha) = \vec{u}\cdot\vec{k}$
$\therefore u_2 = \cos(\alpha) = u_3$,
i.e, $u_2 = u_3$.
$\therefore u_1^2+u_2^2+u_3^2=1 \Rightarrow (\frac{\sqrt{3}}{2})^2+2u_2^2=1 \Rightarrow u_2 = \pm\frac{1}{2\sqrt{2}}$
$\Rightarrow u_3 = \pm\frac{1}{2\sqrt{2}}$.
$\therefore \vec{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right\rangle$ or $\left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right\rangle$.

(6) & (7) $f(x,y) = x^2e^{-xy}$
(a) Let $z = x^2e^{-xy} \Longrightarrow z - x^2e^{-xy} = 0$
$\therefore$ Surface is $F(x,y,z) = z - x^2e^{-xy} = 0$.
At $(x,y) = (1,2)$, $z = 1^2e^{-(1)(2)} = e^{-2}$.
Normal to surface is $\nabla F(1,2,e^{-2})$.
$\nabla F(x,y,z) = \langle -2xe^{-xy} + x^2ye^{-xy}, x^3e^{-xy}, 1\rangle$
$\nabla F(1,2,e^{-2}) = \langle (-2)(1)e^{-(1)(2)} + (1)^2(2)e^{-(1)(2)}, (1)^3e^{-(1)(2)}, 1\rangle$
$= \langle 0, e^{-2}, 1\rangle$.
(b) $\nabla F(1,2,e^{-2}) \cdot \langle x-1, y-2, z-e^{-2}\rangle = 0$
$\Rightarrow e^{-2}(y-2) + 1(z-e^{-2}) = 0 \Rightarrow e^{-2}y + z = 3e^{-2}$

Review Problems 2: Sol^{n}s

(c) $z = x^2 - y^2 \Rightarrow z - x^2 + y^2 = 0$.

Letting $G(x,y,z) = z - x^2 + y^2$, we need the point (x,y,z) at which $\nabla G(x,y,z) = k\langle 0, e^{-2}, 1\rangle$ for some scalar $k \neq 0$.

$\nabla G(x,y,z) = \langle -2x, 2y, 1\rangle$

$\therefore k = 1,\ x = 0,\ y = \frac{1}{2}e^{-2},\ z = x^2 - y^2 = -\frac{1}{4}e^{-4}$

$\therefore$ the point is $(0, \frac{1}{2}e^{-2}, -\frac{1}{4}e^{-4})$.

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(8) Surface is $F(x,y,z) = (\cos x)(\cos y)e^z = 0$

Normal is $\nabla F(\frac{\pi}{2}, 1, 0) = \langle -(\sin x)(\cos y)e^z, -(\cos x)(\sin y)e^z, (\cos x)(\cos y)e^z \rangle \Big|_{(\frac{\pi}{2},1,0)}$

$= \langle -\cos(1), 0, 0\rangle$

$\therefore$ Tangent plane is

$\langle x - \frac{\pi}{2}, y - 1, z - 0\rangle \cdot \langle -\cos(1), 0, 0\rangle = 0$

$\therefore -\cos(1)(x - \frac{\pi}{2}) = 0$

$\therefore x = \frac{\pi}{2}$.

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(9) $x^2 + 2y^2 + 3z^2 = 6$, $P(1,1,1)$.

Particle travels along line through $(1,1,1)$ in direction $\nabla F(1,1,1)$ at 10 units/s, where $F(x,y,z) = x^2 + 2y^2 + 3z^2$.

$\nabla F(1,1,1) = \langle 2x, 4y, 6z\rangle\Big|_{(1,1,1)} = \langle 2, 4, 6\rangle$

$\therefore$ line of travel is

$\vec{r}(t) = \langle 1,1,1\rangle + t\langle 2,4,6\rangle = \langle 1+2t, 1+4t, 1+6t\rangle,\ t \geq 0$.

Since $|\vec{r}\,'(t)| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$, reparametrize the line so that speed is 10: let $t = \frac{10}{\sqrt{56}}s = \frac{5}{\sqrt{14}}s,\ s \geq 0$

$\therefore \vec{r}(s) = \langle 1,1,1\rangle + s\langle \frac{10}{\sqrt{14}}, \frac{20}{\sqrt{14}}, \frac{30}{\sqrt{14}}\rangle$

$= \langle 1 + \frac{10}{\sqrt{14}}s, 1 + \frac{20}{\sqrt{14}}s, 1 + \frac{30}{\sqrt{14}}s\rangle$ $\longrightarrow$

Review Problems 2 : Sol^{n}s

The line intersects the surface $x^2+y^2+z^2=103$.
when $\left(1+\frac{10}{\sqrt{14}}s\right)^2+\left(1+\frac{20}{\sqrt{14}}s\right)^2+\left(1+\frac{30}{\sqrt{14}}s\right)^2=103$.

Solving for s:

$$1+\frac{20}{\sqrt{14}}s+\frac{100}{14}s^2+1+\frac{40}{\sqrt{14}}s+\frac{400}{14}s^2+1+\frac{60}{\sqrt{14}}+\frac{900}{14}s^2=103$$

$$\Rightarrow \quad 100s^2+\frac{120}{\sqrt{14}}s-100=0$$

$$\Rightarrow \quad s=\frac{-\frac{120}{\sqrt{14}}\pm\sqrt{\left(\frac{120}{\sqrt{14}}\right)^2-4(100)(-100)}}{(2)(100)}$$

$$=\frac{1}{200}\left[\frac{-120}{\sqrt{14}}\pm\sqrt{\frac{14400}{14}+40000}\right]$$

$$\approx 0.85,\ \cancel{-1.17}\ \Big\}\text{ since } s\geq 0.$$

$\therefore\ s\approx 0.85\quad\left(s=\frac{1}{70}\left[\sqrt{5026}-3\sqrt{14}\right]\text{ exactly}\right)$.

(10) $f(x,y)=5ye^x-e^{5x}-y^5$ } differentiable on $\mathbb{R}^2$.

$f_x=5ye^x-5e^{5x}$
$f_{xx}=5ye^x-25e^{5x}$
$f_y=5e^x-5y^4$
$f_{yy}=-20y^3$
$f_{xy}=5e^x$

$f_x=f_y=0\Rightarrow$ $5ye^x-5e^{5x}=0$ ①
$5e^x-5y^4=0$ ②

$\Rightarrow$ $5e^x(y-e^{4x})=0$ ③
$5(e^x-y^4)=0$ ④

① $\Rightarrow y=e^{4x}$; sub into ④:
$5(e^x-(e^{4x})^4)=0$
$\Rightarrow e^x-e^{16x}=0$
$\Rightarrow e^x(1-e^{15x})=0$ $\longrightarrow$

Review Problems 2: Sol^{n}s

$\therefore \cancel{e^{x}=0}$ or $e^{15x}=1$

$\therefore x=0 \Rightarrow y=e^{(4)(0)}=1$

$\therefore (0,1)$ is the only C.P.

$D = f_{xx}(0,1)\, f_{yy}(0,1) - \left(f_{xy}(0,1)\right)^2$

$= \left[(5)(1)e^{0} - 25e^{(5)(0)}\right]\left[-20(1)^3\right] - (5e^{0})^2$

$= (5-25e^{5})(-20) - 25$

> 0

and $f_{xx}(0,1) = 5-25e^{5} < 0$,

$\therefore (0,1)$ corresponds to a local maximum.

But, letting $y \to -\infty$ along the y axis (with $x=0$) we see $f(x,y) = 5ye^{x} - e^{5x} - y^{5} \to \infty$.

$\therefore f$ does not have an absolute maximum.