Question 1: Let $f(x, y) = \dfrac{1}{2}\ln(x^2 + y^2) - e^{y/x}$. Calculate $f_x(1, 0) + f_y(1, 0)$

$$f_x(1,0) - f_y(1,0) = \left[\frac{\cancel{2}x}{\cancel{2}(x^2+y^2)} - e^{y/x}\left(-\frac{y}{x^2}\right)\right] + \left[\frac{\cancel{2}y}{\cancel{2}(x^2+y^2)} - e^{y/x}\left(\frac{1}{x}\right)\right]\Bigg|_{(1,0)}$$

$$= \left[\frac{1}{1^2+0^2} - \cancel{e^0\left(\frac{0}{1^2}\right)}^{0}\right] + \left[\cancel{\frac{0}{1^2+0^2}}^{0} - e^0\left(\frac{1}{1}\right)\right]$$

$$= \boxed{0}$$

[4]

Question 2: Let $w(x, y, z) = 4x^3y^2z + \sec(x - \sqrt{1+z^2})$. Detemine w_{xzy}. [You may assume that Clairaut's Theorem applies.]

$W_{xzy} = W_{yxz}$ by Clairaut's Thm

$$= (8x^3yz)_{xz}$$

$$= (24x^2yz)_z$$

$$= \boxed{24x^2y}$$

[3]

Question 3: Let $f(x, y, z) = xy + yz + xz$ where $x = \cos(u^2v)$, $y = \sin(v/u)$ and $z = u + v - 1$. Find $\dfrac{\partial f}{\partial u}$ at the point where $(u, v) = (1, \pi/2)$.

$\dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial u}$ by the chain rule

$$= (y+z)\left(-\sin(u^2v)(2uv)\right) + (x+z)\cos\left(\frac{v}{u}\right)\left(\frac{-v}{u^2}\right) + (x+y)(1)$$

At $(u,v) = (1, \pi/2)$, $x = \cos(1^2 \cdot \frac{\pi}{2}) = 0$, $y = \sin(\pi/2) = 1$, $z = \frac{\pi}{2}$, so

$$\frac{\partial f}{\partial u} = \left(1+\frac{\pi}{2}\right)\left(-\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\left(\cancel{2}\cdot 1 \cdot \frac{\pi}{\cancel{2}}\right)\right) + \left(0+\frac{\pi}{2}\right)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\left(-\frac{\pi}{2}\right) + (0+1)(1)$$

$$= \boxed{1 - \pi\left(1+\frac{\pi}{2}\right)}$$

[3]

Question 4: Determine $\frac{\partial x}{\partial z}$ at the point $(x, y, z) = (1, -1, -3)$ if

$$xz + y\ln(x) - x^2 + 4 = 0$$

$$\frac{\partial}{\partial z}\left[xz + y\ln(x) - x^2 + 4\right] = 0$$

$$\frac{\partial x}{\partial z} z + x\cdot 1 + \frac{y}{x}\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0$$

At $(x,y,z) = (1,-1,-3)$: $\frac{\partial x}{\partial z}(-3) + 1 + \left(\frac{-1}{1}\right)\frac{\partial x}{\partial z} - (2)(1)\frac{\partial z}{\partial z} = 0$

$$-6\frac{\partial x}{\partial z} + 1 = 0 \Rightarrow \boxed{\frac{\partial x}{\partial z} = \frac{1}{6}}$$

[4]

Question 5:

(i) Determine the directional derivative of $f(x, y) = x^2e^{-2y}$ at the point $P(2, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

Here direction as a unit vector is $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$$D_u f(2,0) = \nabla f(2,0)\cdot\vec{u}$$

$$= \langle 2xe^{-2y}, -2x^2e^{-2y}\rangle\Big|_{(2,0)} \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$$

$$= \langle 4, -8\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = \boxed{-4/\sqrt{2}}$$

[2]

(ii) If starting at the point $P(2, 0)$, in which direction in the xy-plane should one proceed so that $f(x, y)$ increases most rapidly? State your answer as a unit vector.

Proceed in direction of $\nabla f(2,0) = \langle 4, -8\rangle$.

As a unit vector: $\frac{\nabla f(2,0)}{|\nabla f(2,0)|} = \frac{\langle 4,-8\rangle}{\sqrt{4^2+8^2}} = \boxed{\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\rangle}$

[2]

(iii) If starting at the point $P(2, 0)$, in which direction in the xy-plane should one proceed so that $f(x, y)$ neither increases nor decreases? State your answer as a unit vector. (There are two possible directions; give one.)

Direction $\langle a, b\rangle$ is such that $\nabla f(2,0)\cdot\langle a,b\rangle = 0$;

$\langle 4,-8\rangle\cdot\langle a,b\rangle = 0 \Rightarrow 4a - 8b = 0 \Rightarrow a - 2b = 0 \Rightarrow a = 2b.$

Since $|\langle a,b\rangle| = 1$, $a^2+b^2 = 1 \Rightarrow (2b)^2 + b^2 = 1 \Rightarrow b = \frac{\pm 1}{\sqrt{5}}$

So directions are $\boxed{\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle, \langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\rangle}$

[2]

Question 6: Use a linear approximation to estimate $f(4.1, -0.2)$ where $f(x, y) = \sqrt{x + \sin(4y)}$.

$(4.1, -0.2)$ is near $(4, 0)$, so

$$f(4.1, -0.2) \approx L(4.1, -0.2)$$
$$= f(4,0) + f_x(4,0)(4.1-4) + f_y(4,0)(-0.2-0)$$
$$= 2 + \left.\frac{1}{2\sqrt{x+\sin(4y)}}\right|_{(4,0)} (0.1) + \left.\frac{4\cos(4y)}{2\sqrt{x+\sin(4y)}}\right|_{(4,0)} (-0.2)$$
$$= 2 + \left(\frac{1}{4}\right)\left(\frac{1}{10}\right) + (1)\left(\frac{-2}{10}\right)$$
$$= \boxed{\frac{73}{40} \text{ or } 1.825}$$

[5]

Question 7: Find all points, if any, on the surface $z^2 - e^{xy} = 3$ at which the tangent plane is horizontal.

Let $F(x,y,z) = z^2 - e^{xy} - 3 = 0$ represent the surface.

If tangent plane is horizontal at $(x,y,z) = (a,b,c)$, then $\nabla F(a,b,c) = \langle 0,0,k \rangle$ for some constant $k \neq 0$.

$$\Rightarrow \langle -ye^{xy}, -xe^{xy}, 2z \rangle_{(a,b,c)} = \langle 0,0,k \rangle$$

$\Rightarrow a = b = 0$ and $2c \neq 0$.

Since $(a,b,c) = (0,0,c)$ is on the surface,

$c^2 - e^0 - 3 = 0$

$\Rightarrow c^2 = 4$

$\Rightarrow c = \pm 2$

$\boxed{\therefore \text{ points are } (0,0,2), (0,0,-2)}$

[5]

Question 8: Find all critical points of $f(x, y) = x^3 + 3xy + y^3$ and classify each as either a local maximum, local minimum or a saddle point. Carefully calculate all required derivatives and keep your work organized.

$f_x = 3x^2 + 3y$, $f_{xx} = 6x$ $\quad f_{xy} = 3$.

$f_y = 3y^2 + 3x$, $f_{yy} = 6y$

$f_x = 0 \Rightarrow y = -x^2$.

$f_y = 0 \Rightarrow x = -y^2 = -(-x^2)^2 = -x^4$

$\Rightarrow x^4 + x = 0$

$\Rightarrow x(x^3+1) = 0$

$\Rightarrow x = 0$, $x = -1$

$x = 0 \Rightarrow y = 0$

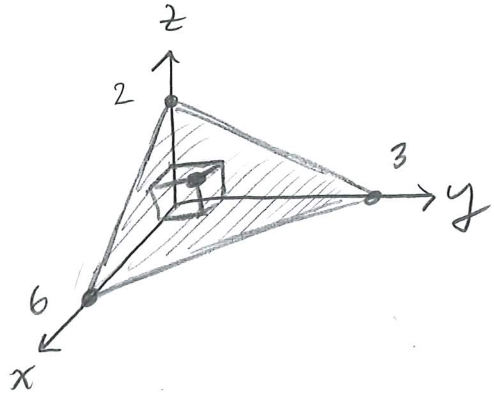
$x = -1 \Rightarrow y = -1$

$\therefore$ CPs are $(0,0)$, $(-1,-1)$.

CP.	$D = f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	Conclusion
$(0,0)$	-9	~	saddle point
$(-1,-1)$	27	$-6 < 0$	loc. max.

[5]

Question 10:
Find the volume of the largest rectangular box in the first octant having three faces on the coordinate planes and one vertex on the plane $x + 2y + 3z = 6$.



Maximize $V = xyz$
subject to $x+2y+3z=6$,
i.e $x = 6-2y-3z$.

Note $x>0$, $y>0$, $z>0$.

So $V = (6-2y-3z)yz$.

$= 6yz - 2y^2z - 3yz^2$

$V_y = 6z - 4yz - 3z^2 = z(6-4y-3z)$

$V_z = 6y - 2y^2 - 6yz = y(6-2y-6z)$

If $V_y = V_z = 0$ with $y>0$ and $z>0$

Then ① $6-4y-3z=0$
② $6-2y-6z=0$

$2\times$②: $6-4y-3z=0$
$-(12-4y-12z)=0$
$-6+9z=0$

$\therefore z = \frac{2}{3}$, and

$6-4y-3z=0$
$\Rightarrow 6-4y-3(\frac{2}{3})=0$
$\Rightarrow y=1$.

Finally, $x = 6-2y-3z$
$\Rightarrow x = 6-2(1)-3(\frac{2}{3})$
$x=2$

$\therefore$ Maximum volume is $V = (2)(1)(\frac{2}{3}) = \frac{4}{3}$.

[10]