Question 1: Find a vector 5 units long in the direction opposite to the direction of $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$.

$$\vec{w} = -5 \frac{\vec{v}}{|\vec{v}|}$$

$$= -5 \frac{\langle 3/5, 0, 4/5 \rangle}{|\langle \frac{3}{5}, 0, \frac{4}{5} \rangle|}$$

$$= -5 \frac{\langle 3/5, 0, 4/5 \rangle}{\sqrt{3/5^2 + 0^2 + 4/5^2}}$$

$$= \boxed{\langle -3, 0, -4 \rangle}$$

[4]

Question 2: For this question use the vectors $\mathbf{u} = -\mathbf{i} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

(i) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos(\theta)$$

$$\Rightarrow \theta = \arccos\left[\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right]$$

$$= \arccos\left[\frac{\langle -1, 0, -1 \rangle \cdot \langle 1, 1, 2 \rangle}{\sqrt{(-1)^2 + 0^2 + (-1)^2}\sqrt{1^2 + 1^2 + 2^2}}\right]$$

$$= \arccos\left[\frac{-3}{2\sqrt{3}}\right]$$

$$= \arccos\left[-\frac{\sqrt{3}}{2}\right]$$

$$= \boxed{\frac{5\pi}{6}}$$

[2]

(ii) Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\text{proj}_{\vec{v}} \vec{u} = \left(\vec{u} \cdot \frac{\vec{v}}{|\vec{v}|}\right) \frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\langle -1, 0, -1 \rangle \cdot \frac{\langle 1, 1, 2 \rangle}{\sqrt{6}}\right) \frac{\langle 1, 1, 2 \rangle}{\sqrt{6}}$$

$$= -\frac{3}{6} \langle 1, 1, 2 \rangle = \boxed{\langle \frac{-1}{2}, \frac{-1}{2}, -1 \rangle}$$

[2]

(iii) Find a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{vmatrix} = \langle 1, 1, -1 \rangle$$

$$\vec{w} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \frac{\langle 1, 1, -1 \rangle}{\sqrt{1^2 + 1^2 + (-1)^2}} = \boxed{\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \rangle}$$

[2]

Question 3: A parallelogram has vertices (corners) at the points $A(2,-1,4)$, $B(1,0,-1)$, $C(1,2,3)$ and $D(2,1,8)$.

(i) Find the area of the parallelgram.

$$\text{Area} = |\vec{AB} \times \vec{AC}| = \left| \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -5 \\ -1 & 3 & -1 \end{bmatrix} \right| = |\langle 14, 4, -2 \rangle| = \boxed{6\sqrt{6}}$$

[2]

(ii) Find an equation of the plane containing the parallelogram.

From (i), normal vector is $\frac{1}{2}\langle 14,4,-2\rangle = \langle 7,2,-1\rangle$

Using point $A(2,-1,4)$:

$$(\langle x,y,z\rangle - \langle 2,-1,4\rangle)\cdot\langle 7,2,-1\rangle = 0$$

$$\Rightarrow 7(x-2) + 2(y+1) - (z-4) = 0$$

$$\Rightarrow \boxed{7x+2y-z=8}$$

[3]

Question 4: Find the point in which the line through the origin perpendicular to the plane $2x-y-z=4$ intersect the plane $3x-5y+2z=6$.

$2x-y-z=4$ has normal $\langle 2,-1,-1\rangle$.

Line through origin $\perp$ to $2x-y-z=4$

has equation $\vec{r}(t) = \langle 0,0,0\rangle + t\langle 2,-1,-1\rangle = \langle 2t,-t,-t\rangle$.

Line intersects plane $3x-5y+2z=6$

$$\Rightarrow 3(2t) - 5(-t) + 2(-t) = 6$$

$$\Rightarrow 9t = 6$$

$$\Rightarrow t = \frac{2}{3}.$$

$\therefore$ Point of intersection given by $\vec{r}\left(\frac{2}{3}\right) = \left\langle \frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right\rangle$

[5]

so point is $\boxed{P\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)}$

Question 5: Find an equation of the line through the point $(0, 14, -10)$ that is parallel to the line $x = -1+2t$, $y = 6 - 3t$, $z = 3 + 9t$.

$\langle x, y, z\rangle = \langle -1, 6, 3\rangle + t\langle 2, -3, 9\rangle$ has direction vector $\langle 2, -3, 9\rangle$.

So line through $(0, 14, -10)$ with the same direction has equation

$$\boxed{\vec{r}(t) = \langle 0, 14, -10\rangle + t\langle 2, -3, 9\rangle}$$

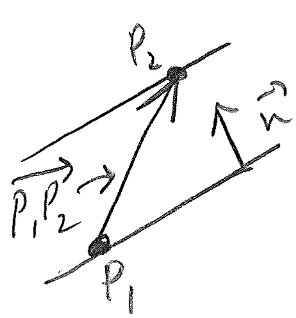
[5]

Question 6: Find the (shortest) distance between the parallel planes $x - 2y + 3z = 1$ and $x - 2y + 3z = 4$.

Point on 1st plane is $P_1(1, 0, 0)$,

Point on 2nd plane is $P_2(4, 0, 0)$.

Normal to both planes is $\vec{n} = \langle 1, -2, 3\rangle$.



Distance between planes is then

$$\left|\text{comp}_{\vec{n}} \overrightarrow{P_1P_2}\right| = \overrightarrow{P_1P_2} \cdot \frac{\vec{n}}{|\vec{n}|}$$

$$= \left| \langle 3, 0, 0\rangle \cdot \frac{\langle 1, -2, 3\rangle}{|\langle 1, -2, 3\rangle|} \right|$$

$$= \boxed{\frac{3}{\sqrt{14}}}$$

[5]

Question 7: A force of magnitude 10 N acts directly upward from the xy-plane on an object of mass 2 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$. Find $\mathbf{r}(1)$, its position after one second.

$$\vec{F} = m\vec{a}$$

$$10\hat{k} = 2\vec{a}$$

$$\Rightarrow \vec{a} = 5\hat{k} = \langle 0,0,5\rangle, \quad \vec{v}(0) = \langle 1,1,0\rangle, \quad \vec{r}(0) = \langle 0,0,0\rangle.$$

$$\therefore \vec{v}(t) = \langle 0,0,5t\rangle + \vec{C_1}$$

$$\vec{v}(0) = \langle 0,0,5\cdot 0\rangle + \vec{C_1} = \langle 1,1,0\rangle \Rightarrow \vec{C_1} = \langle 1,1,0\rangle$$

$$\therefore \vec{v}(t) = \langle 1,1,5t\rangle$$

$$\therefore \vec{r}(t) = \langle t, t, \tfrac{5}{2}t^2\rangle + \vec{C_2}$$

$$\vec{r}(0) = 0 \Rightarrow \vec{C_2} = \vec{0}.$$

$$\therefore \vec{r}(t) = \langle t, t, 5t^2/2\rangle$$

and $\boxed{\vec{r}(1) = \langle 1, 1, 5/2\rangle}$

[5]

Question 8: Determine the length of the curve $\mathbf{r}(t) = \langle \frac{2t^{3/2}}{3}, \cos(2t), \sin(2t)\rangle$ for $0 \le t \le 5$.

$$L = \int_{t=0}^{5} |\vec{r}\,'(t)|\, dt$$

$$= \int_0^5 \sqrt{(t^{1/2})^2 + (-2\sin(2t))^2 + (2\cos(2t))^2}\, dt$$

$$= \int_0^5 (t+4)^{1/2}\, dt$$

$$= \frac{2}{3}(t+4)^{3/2}\Big|_0^5 = \frac{2}{3}(27-8) = \boxed{\frac{38}{3}}$$

[5]

Question 9: For the space curve $\mathbf{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)\rangle$ find the following at $t = \pi$:

(i) The unit tangent vector $\mathbf{T}(\pi)$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$= \frac{\langle 2t, \cancel{\cos(t)} - \cancel{\cos(t)} + t\sin(t), -\cancel{\sin(t)} + \cancel{\sin(t)} + t\cos(t)\rangle}{|\vec{r}'(t)|}$$

$$= \frac{\langle 2t, t\sin(t), t\cos(t)\rangle}{\sqrt{(2t)^2 + t^2\sin^2(t) + t^2\cos^2(t)}}$$

$$= \left\langle \frac{2}{\sqrt{5}}, \frac{\sin(t)}{\sqrt{5}}, \frac{\cos(t)}{\sqrt{5}} \right\rangle$$

so

$$\boxed{\vec{T}(\pi) = \left\langle \frac{2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}} \right\rangle}$$

[2]

(ii) The unit normal vector $\mathbf{N}(\pi)$.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\left\langle 0, \frac{\cos(t)}{\sqrt{5}}, \frac{-\sin(t)}{\sqrt{5}} \right\rangle}{\frac{1}{\sqrt{5}}} = \langle 0, \cos(t), -\sin(t)\rangle$$

$$\boxed{\therefore \ \vec{N}(\pi) = \langle 0, -1, 0\rangle}$$

[2]

(iii) The curvature κ.

$$k = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\left|\left\langle 0, \frac{\cos(t)}{\sqrt{5}}, \frac{-\sin(t)}{\sqrt{5}} \right\rangle\right|}{|\sqrt{5}\, t|}$$

so at $t = \pi$:

$$k = \frac{\left|\left\langle 0, \frac{-1}{\sqrt{5}}, 0 \right\rangle\right|}{|\sqrt{5}\,\pi|} = \boxed{\frac{1}{5\pi}}$$

[2]