Question 1: Eliminate the parameter to find a Cartesian equation of the following parametric curve, then sketch the curve for the given range of parameter values. Indicate with an arrow the direction in which the curve is traced as the t parameter increases:

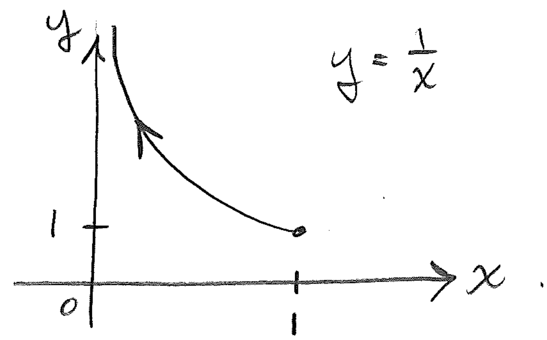
$$x = \cos(t), \quad y = \sec(t), \quad 0 \le t < \pi/2$$

$y = \sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$

As $t: 0 \to \frac{\pi}{2}^-$,

$x : 1 \to 0^+$,

so $y : 1 \to \infty$.



[5]

Question 2: For this question use the parametric curve $x = t^3 - 12t + 1$, $y = 2t - t^2$.

(i) Find an equation of the tangent line to the curve at the point where $t = -1$

$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2-2t}{3t^2-12}$

At $t = -1$, $x = (-1)^3 - 12(-1) + 1 = 12$

$y = 2(-1) - (-1)^2 = -3$

$\frac{dy}{dx} = \frac{2-2(-1)}{3(-1)^2-12} = \frac{-4}{9}$

$\therefore$ Equation of tangent line is

$\boxed{y+3 = -\frac{4}{9}(x-12)}$

[3]

(ii) Find all points (x, y) along the curve at which tangent lines are vertical.

Require $\frac{dx}{dt} = 0$ while $\frac{dy}{dt} \neq 0$.

$\frac{dx}{dt} = 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow 3(t-2)(t+2) = 0 \Rightarrow t = 2, -2.$

Note that $\left.\frac{dy}{dt}\right|_{t=2} \neq 0$, $\left.\frac{dy}{dt}\right|_{t=-2} \neq 0$

At $t = 2$ $(x,y) = (-15, 0)$

$t = -2$ $(x,y) = (17, -8)$

$\boxed{\therefore \text{Points are } (-15,0), (17,-8)}$

[2]

Question 3: Determine the length of the parametric curve:

$$x = \frac{t^2}{2},\quad y = \frac{(2t+1)^{3/2}}{3},\quad 0 \le t \le 4$$

$$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^4 \sqrt{\left(\frac{2t}{2}\right)^2 + \left[\frac{3}{2}\,\frac{(2t+1)^{1/2}\cdot 2}{3}\right]^2}\, dt$$

$$= \int_0^4 \sqrt{t^2 + 2t + 1}\, dt$$

$$= \int_0^4 \sqrt{(t+1)^2}\, dt$$

$$= \int_0^4 (t+1)\, dt$$

$$= \left[\frac{t^2}{2} + t\right]_0^4$$

$$= \left(\frac{4^2}{2} + 4\right) - 0$$

$$= \boxed{12}$$

[5]

Question 4: Determine the area of the region between the parametric curve and the y-axis:

$$x = e^{-t} + 1,\quad y = e^{2t} - 1,\quad 0 \le t \le 1$$

For $t: 0 \to 1$ y increases from 0 to $e^2 - 1$ and $x > 0$.

$$\therefore\ A = \int_{y=0}^{e^2-1} x\, dy$$

$$= \int_{t=0}^{1} (e^{-t}+1)\, \frac{d}{dt}(e^{2t}-1)\, dt$$

$$= \int_0^1 (e^{-t}+1)(2e^{2t})\, dt$$

$$= 2\int_0^1 (e^{t} + e^{2t})\, dt$$

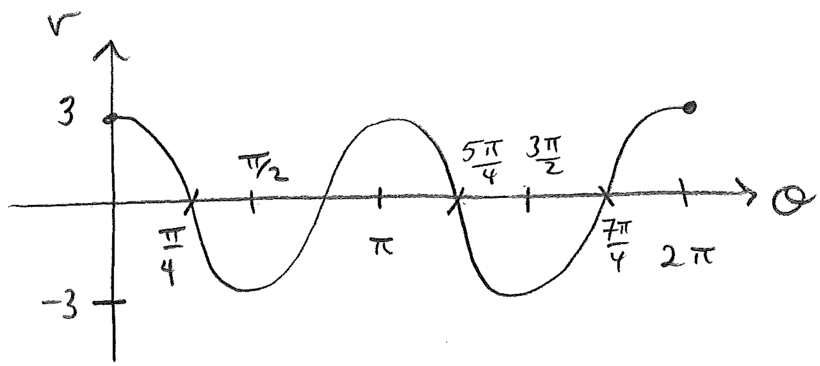
$$= 2\left[e^t + \frac{e^{2t}}{2}\right]_0^1$$

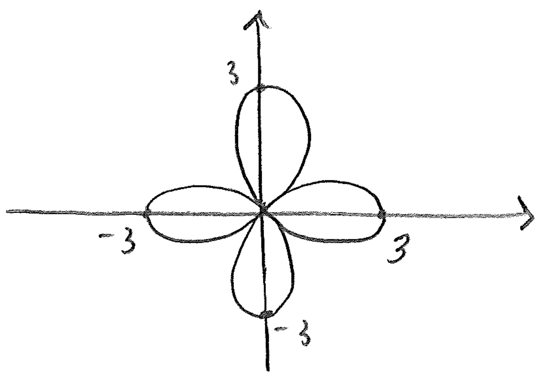
$$= 2\left[\left(e + \frac{e^2}{2}\right) - \left(1 + \frac{1}{2}\right)\right]$$

$$= \boxed{2e + e^2 - 3}$$

[5]

Question 5: Neatly sketch the polar curve $r = 3\cos(2\theta)$.





[5]

Question 6: Find an equation (in Cartesian coordinates) for the tangent line to $r = 3\cos(\theta)$ at the point where $\theta = \pi/6$.

At $\theta = \frac{\pi}{6}$: $x = r\cos\theta = 3\cos(\theta)\cos(\theta) = 3\cos^2\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4}$

$y = r\,\text{sen}\,\theta = 3\cos(\theta)\,\text{sen}(\theta) = 3\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{4}$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-3\sin(\theta)\,\text{sen}(\theta) + 3\cos(\theta)\cos(\theta)}{-3\sin(\theta)\cos(\theta) - 3\cos(\theta)\sin(\theta)}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{-2\sin\theta\cos\theta}$$

so $$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{\cos^2(\pi/6) - \sin^2(\pi/6)}{-2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)} = \frac{\left(\sqrt{3}/2\right)^2 - \left(\frac{1}{2}\right)^2}{-\cancel{2}\left(\frac{1}{\cancel{2}}\right)\left(\frac{\sqrt{3}}{2}\right)} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}$$

Equation of tangent line is $\boxed{y - \frac{3\sqrt{3}}{4} = \frac{-1}{\sqrt{3}}\left(x - \frac{9}{4}\right)}$

[5]

Question 7: Find the area enclosed by one loop of the polar curve $r = 3\cos(2\theta)$ from question 5.

There are 4 loops of equal area, so area of one

is $A = \frac{1}{4}\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$

$$= \frac{1}{4}\int_0^{2\pi} \frac{1}{2}\left[3\cos(2\theta)\right]^2 d\theta$$

$$= \frac{9}{8}\int_0^{2\pi} \cos^2(2\theta)\, d\theta$$

$$= \frac{9}{8}\int_0^{2\pi} \frac{1+\cos(4\theta)}{2}\, d\theta$$

$$= \frac{9}{16}\left[\theta + \frac{\sin(4\theta)}{4}\right]_0^{2\pi}$$

$$= \frac{9}{16}\left[(2\pi+0)-(0+0)\right]$$

$$= \boxed{\frac{9\pi}{8}}$$

[5]

Question 8: Find the arc length of $r = 3\cos(\theta)$ over $0 \le \theta \le \pi/6$.

$$L = \int_0^{\frac{\pi}{6}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{9\cos^2\theta + 9\sin^2\theta}\, d\theta$$

$$= 3\int_0^{\pi/6} 1 \, d\theta$$

$$= (3)\left(\frac{\pi}{6}\right)$$

$$= \boxed{\frac{\pi}{2}}$$

[5]

Question 9: For this question use the sphere $x^2 + (y-3)^2 + (z+5)^2 = 4$:

(i) Find the point on the sphere that is closest to the xy-plane.

Sphere has centre $(0,3,-5)$ and radius 2, so is located below the xy-plane. The point on the sphere closest to the xy plane is therefore a distance 2 directly above the centre, so has coordinates $(0,3,-5+2) = \boxed{(0,3,-3)}$

[2]

(ii) Find the point on the sphere that is closest to the point $(0, 7, -5)$.

The distance from $(0,7,-5)$ to the centre $(0,3,-5)$ is 4, So $(0,7,-5)$ is outside of the sphere.

Since $(0,7,-5)$ has the same x & z coordinates as the centre, the nearest point on the sphere will be found in a direction parallel to the positive y-axis and a distance 2 from the centre, so has coordinates $(0,3+2,-5) = \boxed{(0,5,-5)}$

[3]

Question 10: For this question use the vectors

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

(i) Compute $|3\mathbf{a} - 2\mathbf{b} - \mathbf{c}| = |3\langle 2,-1,3\rangle - 2\langle 3,-2,1\rangle - \langle 1,1,1\rangle|$

$$= |\langle -1,0,6\rangle|$$

$$= \sqrt{(-1)^2 + 0^2 + 6^2}$$

$$= \boxed{\sqrt{37}}$$

[2]

(ii) Find a vector of magnitude 5 pointing in the same direction as $(\mathbf{b} - \mathbf{c})$.

The required vector is $\vec{v} = 5 \dfrac{\vec{b}-\vec{c}}{|\vec{b}-\vec{c}|}$

$$= 5 \frac{\langle 3,-2,1\rangle - \langle 1,1,1\rangle}{|\langle 3,-2,1\rangle - \langle 1,1,1\rangle|}$$

$$= \frac{\langle 10,-15,0\rangle}{\sqrt{2^2+(-3)^2+0^2}} = \boxed{\left\langle \frac{10}{\sqrt{13}}, \frac{-15}{\sqrt{13}}, 0 \right\rangle}$$

[3]