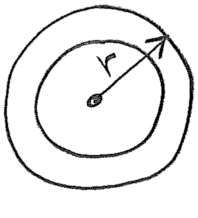
Question 1: For this question use the fact that a sphere (ball) of radius r has volume $V = \frac{4}{3}\pi r^3$ and outer surface area $S = 4\pi r^2$.

An iron ball of radius 4 cm is coated with a layer of ice of uniform thickness and the ice is melting at a rate of 10 cm^3/min.

(a) How fast is the thickness of the ice decreasing when it is 2 cm thick?



Let $r(t)$ = radius of ball together with ice.

$V(t)$ = volume of ball together with ice.

$$\frac{dV}{dt} = -10 \frac{cm^3}{min}.$$

Find $\frac{dr}{dt}$ when $r = 4+2 = 6$ cm.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$$

when $r=6$:

$$-10 = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} \cdot 6^2 \frac{dr}{dt}$$

So

$$\frac{dr}{dt} = \frac{-10}{(4)(36)\pi} = \frac{-5}{72\pi}$$

$\boxed{\therefore \text{ ice thickness is decreasing at } \frac{5}{72\pi} \text{ cm/min}}$

[6]

(b) How fast is the outer surface area the ice decreasing when it is 2 cm thick?

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 4\pi\, 2r \frac{dr}{dt}$$

when $r = 6$:

$$\frac{dS}{dt} = (\cancel{4}^{2}\cancel{\pi})(\cancel{2})(\cancel{6})\left(\frac{-5}{\cancel{72}_{3}\,_{6}\cancel{\pi}}\right) = \frac{-10}{3}$$

$\boxed{\text{So surface area is decreasing at } \frac{10}{3} \frac{cm^2}{min}}$

[4]

Question 2: Find the following derivatives (Take care to use proper notation; it is not necessary to simplify your answers):

(a) $f(x) = \sqrt{x}\,[\ln(x)]^2$ $= x^{1/2}[\ln(x)]^2$

$$f'(x) = \boxed{\frac{1}{2}x^{-\frac{1}{2}}[\ln(x)]^2 + x^{1/2}\cdot 2\ln(x)\cdot\frac{1}{x}}$$

[2]

(b) $y = 10e^{-5/x}$

$$y' = \boxed{10e^{-5/x}\cdot\frac{5}{x^2}}$$

$$= \frac{50e^{-5/x}}{x^2}$$

[2]

(c) $g(t) = \log_2(4^{\sec(t)-t})$ $= (\sec(t)-t)\log_2(4) = 2(\sec(t)-t)$

$$\boxed{\therefore\ g'(t) = 2(\sec(t)\tan(t)-1)}$$

[3]

Question 3: Find ~~an equation~~ the slope of the tangent line to the curve $e^{y/x} = x^3y + x - 2$ at the point $(x, y) = (2, 0)$.

$$\frac{d}{dx}[e^{y/x}] = \frac{d}{dx}[x^3y + x - 2]$$

$$e^{y/x}\cdot\left[\frac{xy'-y}{x^2}\right] = 3x^2y + x^3y' + 1$$

at $(x,y) = (2,0)$:

$$\underbrace{e^{0/2}}_{1}\left[\frac{2y'-0}{2^2}\right] = 3\cancel{(2^2)}(0) + 2^3y' + 1$$

$$\frac{y'}{2} = 8y' + 1$$

so $\boxed{y' = \frac{-2}{15}}$.

[3]

Question 4: Use a linear approximation to estimate $(2.2)^3$.

Let $f(x) = x^3$, $a = 2$.

$f(a) = f(2) = 8$.

$f'(x) = 3x^2$;

$f'(a) = f'(2) = 12$.

$\therefore$ $L(x) = f(a) + f'(a)(x-a)$

$= 8 + 12(x-2)$.

$\therefore$ $(2.2)^3 = f(2.2) \approx L(2.2)$

$= 8 + 12(2.2-2)$

$= 8 + 2.4$

$= \boxed{10.4}$

[5]

Question 5: Find an equation of the tangent line to $y = x^{\sin(x)}$ at the point where $x = \frac{\pi}{2}$.
[Hint: logarithmic differentiation]

At $x = \frac{\pi}{2}$, $y = \left(\frac{\pi}{2}\right)^{\sin(\pi/2)} = \frac{\pi}{2}$.

For y': $\ln(y) = \ln\left(x^{\sin(x)}\right)$

$\Rightarrow \ln(y) = \sin(x)\ln(x)$.

$\Rightarrow \frac{1}{y} y' = \cos(x)\ln(x) + \sin(x)\frac{1}{x}$

$y' = x^{\sin(x)}\left[\cos(x)\ln(x) + \sin(x)\frac{1}{x}\right]$

At $x = \frac{\pi}{2}$: $y' = \left(\frac{\pi}{2}\right)\left[\cancel{\cos\left(\frac{\pi}{2}\right)}^{0}\ln\left(\frac{\pi}{2}\right) + \cancel{\sin\left(\frac{\pi}{2}\right)}^{1}\left(\frac{1}{(\pi/2)}\right)\right]$

$= 1$

$\therefore$ Equation is $y - \frac{\pi}{2} = 1\cdot\left(x - \frac{\pi}{2}\right)$ or $\boxed{y = x}$

[5]

Question 6: Evaluate the following limits, if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning.

(a) $\lim_{x\to 0} \frac{\cos(x)-1}{x^2} \sim \text{"}\frac{0}{0}\text{"}$

$$\overset{H}{=} \lim_{x\to 0} \frac{-\sin(x)}{2x} \sim \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0} -\frac{\cos(x)}{2}$$

$$= \boxed{-\frac{1}{2}}$$

[3]

(b) $\lim_{x\to\infty} \frac{e^x - e^{-x} + 2x}{x^2 + \ln(x)} \sim \text{"}\frac{\infty}{\infty}\text{"}$

$$\overset{H}{=} \lim_{x\to\infty} \frac{e^x + e^{-x} + 2}{2x + \frac{1}{x}} \sim \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{e^x - e^{-x}}{2 - \frac{1}{x^2}} = \boxed{\infty}$$

[3]

(c) $\lim_{x\to 0^+} x[\ln(x)]^2 \sim \text{"}0 \cdot \infty\text{"}$

$$= \lim_{x\to 0^+} \frac{[\ln(x)]^2}{\left(\frac{1}{x}\right)} \sim \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{2\ln(x)\cancel{\left(\frac{1}{x}\right)}}{-\left(\frac{1}{x}\right)^{\cancel{2}}} \sim \text{"}\frac{-\infty}{-\infty}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{2\cancel{\left(\frac{1}{x}\right)}}{\left(\frac{1}{x}\right)^{\cancel{2}}} = \boxed{0}$$

[4]

Question 7: Determine the absolute minimum and maximum values of $f(x) = \dfrac{x}{x^2+1}$ on the interval $[-3, 4]$.

Domain $(-\infty, \infty)$.

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

- $f'(x)=0$? $x = 1, -1$.
- $f'(x)$ not exist? no such x.

x	$f(x) = \frac{x}{x^2+1}$	
-3	$-\frac{3}{10}$	
-1	$-\frac{1}{2}$	← abs. min
1	$\frac{1}{2}$	← abs. max.
4	$\frac{4}{17}$	

$\therefore$ f has an abs. max. of $\frac{1}{2}$ at $x=1$;
f has an abs. min. of $-\frac{1}{2}$ at $x=-1$

[5]

Question 8: Determine the intervals of increase and decrease of $f(x) = x^2e^{-x}$. Domain is $(-\infty, \infty)$

$$f'(x) = 2xe^{-x} + x^2e^{-x}(-1) = xe^{-x}(2-x)$$

- $f'(x) = 0$? $x = 0, 2$
- $f'(x)$ not exist? no such x.

Crit. nums:		0		2	
test. pts.:	-1		1		3
$f'(x) = x(2-x)\underbrace{e^{-x}}_{\text{always } \oplus}$:	$-$		$+$		$-$
$f(x)$:	↘		↗		↘

$\therefore$ f decreasing on $(-\infty,0)\cup(2,\infty)$, increasing on $(0,2)$

[5]