Question 1: Evaluate the following limits, if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning. (Do not use L'Hospital's rule to evaluate limits.)

(a) $\lim_{x\to 0} \frac{3x+1}{x^2(x+1)}$ $\begin{matrix} \} \to 1 \\ \} \to 0^+ \end{matrix}$

$= \boxed{+\infty}$

[2]

(b) $\lim_{x\to -4} \frac{2x+8}{|x+4|}$: • As $x \to -4^+$, $x+4>0$, so $|x+4| = x+4$.

$\therefore \lim_{x\to -4^+} \frac{2x+8}{|x+4|} = \lim_{x\to -4^+} \frac{2\cancel{(x+4)}}{\cancel{(x+4)}} = 2$

• As $x \to -4^-$, $x+4<0$, so $|x+4| = -(x+4)$.

$\therefore \lim_{x\to -4^-} \frac{2x+8}{|x+4|} = \lim_{x\to -4^-} \frac{2\cancel{(x+4)}}{-\cancel{(x+4)}} = -2$

Since $\lim_{x\to -4^+} \frac{2x+8}{|x+4|} \neq \lim_{x\to -4^-} \frac{2x+8}{|x+4|}$,

the general limit $\lim_{x\to -4} \frac{2x+8}{|x+4|}$ $\boxed{\text{does not exist}}$

[4]

Question 2: Decide whether the following function is continuous ~~for all real numbers~~ at $x=4$. Explain your reasoning.

$$f(x) = \begin{cases} \dfrac{x^2-4x}{x^2-3x-4} & \text{if } x \neq 4 \\ 5/4 & \text{if } x = 4 \end{cases}$$

$\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4} \frac{x\cancel{(x-4)}}{(x+1)\cancel{(x-4)}} = \frac{4}{5} \neq f(4)$,

Since $\lim_{x\to 4} f(x) \neq f(4)$, f is not continuous at $x=4$.

[4]

Question 3:

(a) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{x+1}$. Neatly show all steps and use proper notation. (No credit will be given if $f'(x)$ is found using derivative rules.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \boxed{\frac{1}{2\sqrt{x+1}}}$$

[6]

(b) Find an equation of the tangent line to the graph of $y = \sqrt{x+1}$ at the point where $x = 8$.

At $x = 8$, $y = \sqrt{8+1} = 3$.

Slope of tangent line is $f'(8) = \dfrac{1}{2\sqrt{8+1}} = \dfrac{1}{6}$.

$\therefore$ Equation is

$$\boxed{y - 3 = \frac{1}{6}(x-8)}$$

[4]

Question 4: Find the following derivatives (it is not necessary to simplify your answers, but marks will be deducted for improper use of notation):

(a) $f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x}} = \dfrac{x^{1/2}}{1+x^{1/2}}$

$$f'(x) = \boxed{\frac{(1+x^{1/2})\frac{1}{2}x^{-1/2} - x^{1/2}\cdot(\frac{1}{2})x^{-1/2}}{(1+x^{1/2})^2}}$$

$$= \frac{1}{2\sqrt{x}\,(1+\sqrt{x})^2}$$

[2]

(b) $y = (t^3-3)(2\tan(t)+t)$

$$\boxed{y' = 3t^2(2\tan(t)+t) + (t^3-3)(2\sec^2(t)+1)}$$

[3]

(c) $g(x) = \dfrac{\sec(x)}{1+x+x^2}$

$$\boxed{g'(x) = \frac{(1+x+x^2)\sec(x)\tan(x) - \sec(x)(1+2x)}{(1+x+x^2)^2}}$$

[3]

(d) $y = \underbrace{3\sin(\pi)}_{} - \underbrace{2\cos(\pi)}_{} - \dfrac{\pi^2}{x}$

Constants

$$y' = 0 - 0 - \pi^2(-1)x^{-2}$$

$$= \boxed{\frac{\pi^2}{x^2}}$$

[2]

Question 5: Find the following derivatives (it is not necessary to simplify your answers, but marks will be deducted for improper use of notation):

(a) $y = (x^2 + \csc(x) + 1)^3$

$$\boxed{y' = 3\left(x^2 + \csc(x) + 1\right)^2 \left(2x - \csc(x)\cot(x)\right)}$$

[2]

(b) $y = \cos^4(1 - 2t)$ $= \left[\cos(1-2t)\right]^4$

$$y' = \boxed{4\cos^3(1-2t) \cdot \left[-\sin(1-2t)\right] \cdot (-2)}$$

$$= 8\cos^3(1-2t)\sin(1-2t)$$

[3]

(c) $g(x) = \sqrt{\dfrac{x^3 + x}{x^5}}$ $= \left(\dfrac{x^3+x}{x^5}\right)^{1/2}$

$$\boxed{g'(x) = \frac{1}{2}\left(\frac{x^3+x}{x^5}\right)^{-1/2}\left[\frac{x^5(3x^2+1) - (x^3+x)(5x^4)}{x^{10}}\right]}$$

[3]

(d) $y = \sqrt[3]{x}\,[\sin(x) - \cos(x)]^2$ $= x^{1/3}\left[\sin(x) - \cos(x)\right]^2$

$$\boxed{y' = \frac{1}{3}x^{-2/3}\left[\sin(x) - \cos(x)\right]^2 + x^{1/3} \cdot 2\left[\sin(x) - \cos(x)\right] \cdot \left[\cos(x) + \sin(x)\right]}$$

[2]

Question 6: A rock thrown vertically from the surface of the moon reaches a height of

$$s(t) = 24t - kt^2 \text{ metres}$$

in t seconds where k is a constant. The acceleration experienced by the rock is a constant -1.6 m/s^2. How long does it take for the rock to reach a velocity equal to half of its initial value?

Want t such that $v(t) = \frac{1}{2} v(0)$.

$v(t) = s'(t) = 24 - 2kt$

To find k, we are given that $a(t) = -1.6$ at each t,

so $v'(t) = -1.6$

$\Rightarrow 0 - 2k = -1.6$

so $k = \frac{-1.6}{-2} = 0.8 = \frac{4}{5}$.

$\therefore v(t) = 24 - 2\left(\frac{4}{5}\right)t = 24 - \frac{8}{5}t$.

Solving $v(t) = \frac{1}{2} v(0)$: $24 - \frac{8}{5}t = \frac{1}{2}\left(24 - \frac{8}{5}(0)\right)$

$\Rightarrow \boxed{t = 15/2 \text{ s}}$

[5]

Question 7: Find an equation of the tangent line to the curve defined by $y = 2\sin(x - y)$ at the point $(x, y) = (\pi, 0)$.

$\frac{dy}{dx} = \frac{d}{dx}\left[2\sin(x-y)\right]$

$y' = 2\cos(x-y)(1-y')$

At $(x,y) = (\pi, 0)$:

$y' = 2\cos(\pi - 0)\cdot(1-y')$

$y' = -2(1-y')$

$\Rightarrow y' = 2$

So equation is

$y - 0 = 2(x - \pi)$

or

$\boxed{y = 2(x-\pi)}$

or

$y = 2x - 2\pi$.

[5]