Review Problems 3: Sol^{n}s

(1) Place the molecule with one hydrogen atom at $P_1(0,0,1)$ and a second at $P_2(x_2, 0, z_2)$. The third and fourth hydrogen atoms then have positions $P_3(x_3, y_3, z_2)$, $P_4(x_4, y_4, z_2)$.

Now $|\vec{OP_1}| = |\vec{OP_2}| = |\vec{OP_3}| = |\vec{OP_4}| = 1$.

$\vec{OP_1} \cdot \vec{OP_2} = |\vec{OP_1}| \cdot |\vec{OP_2}| \cos(\theta)$ where θ is the common bond angle.

$\therefore \langle 0,0,1 \rangle \cdot \langle x_2, 0, z_2 \rangle = z_2 = \cos\theta$.

Since $|\vec{OP_2}| = 1$, $x_2 = \sqrt{1 - z_2^2} = \sin\theta$.

$\therefore P_2$ is $(\sin\theta, 0, \cos\theta)$.

Now project P_2, P_3 & P_4 onto the xy-plane resulting in points $Q_2(\sin\theta, 0, 0)$, $Q_3(x_3, y_3, 0)$ & $Q_4(x_4, y_4, 0)$, respectively; The common angle between $\vec{OQ_2}$, $\vec{OQ_3}$ & $\vec{OQ_4}$ is $120°$ and the common magnitude is $|\vec{OQ_2}| = \sin\theta$

$\therefore \vec{OQ_2} \cdot \vec{OQ_3} = |\vec{OQ_2}|\,|\vec{OQ_3}| \cos(120°)$

$\Rightarrow x_3 \sin\theta = \sin^2\theta\left(-\frac{1}{2}\right)$, $\sin\theta \neq 0$

$\Rightarrow x_3 = -\frac{\sin\theta}{2}$

$\Rightarrow y_3 = \left[1 - x_3^2 - z_3^2\right]^{1/2} = \left[1 - \left(-\frac{\sin\theta}{2}\right)^2 - (\cos\theta)^2\right]^{1/2} = \frac{\sqrt{3}}{2}\sin\theta$

$\therefore P_3$ is $P_3\left(-\frac{\sin\theta}{2}, \frac{\sqrt{3}}{2}\sin\theta, \cos\theta\right)$.

Finally, $\vec{OP_2} \cdot \vec{OP_3} = \cos\theta$

$\Rightarrow \langle \sin\theta, 0, \cos\theta \rangle \cdot \left\langle -\frac{\sin\theta}{2}, \frac{\sqrt{3}}{2}\sin\theta, \cos\theta \right\rangle = \cos(\theta)$

$\Rightarrow -\frac{\sin^2\theta}{2} + \cos^2\theta = \cos\theta$

$\Rightarrow \frac{\cos^2\theta - 1}{2} + \cos^2\theta = \cos\theta$

$\Rightarrow 3\cos^2\theta - 2\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)}$

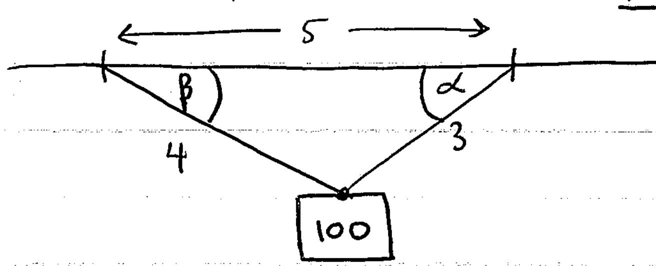
$= \cancel{1}, -\frac{1}{3}$ since $0 < \theta < 180$

$\therefore \theta = \arccos\left(\frac{-1}{3}\right)$

$\approx \boxed{109.47°}$

Review Problems 3: sol^{n}s

(2)



Let $\vec{T}_1$ = tension force in 3 ft rope,
$\vec{T}_2$ = tension force in 5 ft rope.

$\vec{T}_1 = |\vec{T}_1|\cos(\alpha)\hat{\imath} + |\vec{T}_1|\sin(\alpha)\hat{\jmath}$

$\vec{T}_2 = -|\vec{T}_2|\cos(\beta)\hat{\imath} + |\vec{T}_2|\sin(\beta)\hat{\jmath}$.

$\vec{T}_1 + \vec{T}_2 = 100\hat{\jmath}$.

For α & β use law of cosines:

$$4^2 = 3^2 + 5^2 - (2)(3)(5)\cos(\alpha)$$

$$\therefore \cos(\alpha) = \frac{4^2 - 3^2 - 5^2}{-(2)(3)(5)} = \frac{3}{5} \Rightarrow \alpha = \cos^{-1}\left(\frac{3}{5}\right)$$

$$\cos(\beta) = \frac{3^2 - 4^2 - 5^2}{-(2)(4)(5)} = \frac{4}{5} \Rightarrow \beta = \cos^{-1}\left(\frac{4}{5}\right).$$

and $\sin(\alpha) = \sqrt{1 - \cos^2(\alpha)} = \frac{4}{5}$

$\sin(\beta) = \sqrt{1 - \cos^2(\beta)} = \frac{3}{5}$

$\therefore \vec{T}_1 = |\vec{T}_1| \cdot \left(\frac{3}{5}\right)\hat{\imath} + |\vec{T}_1| \cdot \left(\frac{4}{5}\right)\hat{\jmath}$

$\vec{T}_2 = -|\vec{T}_2| \cdot \left(\frac{4}{5}\right)\hat{\imath} + |\vec{T}_2|\left(\frac{3}{5}\right)\hat{\jmath}$

$\therefore \vec{T}_1 + \vec{T}_2 = 100\hat{\jmath} \Rightarrow |\vec{T}_1|\left(\frac{3}{5}\right) - |\vec{T}_2|\left(\frac{4}{5}\right) = 0$ ①

$|\vec{T}_1|\left(\frac{4}{5}\right) + |\vec{T}_2|\left(\frac{3}{5}\right) = 100$ ②

3① + 4② $\Rightarrow 5|\vec{T}_1| = 400 \Rightarrow |\vec{T}_1| = 80$

① $\Rightarrow |\vec{T}_2| = \frac{3}{4}|\vec{T}_1| = 60$.

$$\boxed{\begin{array}{ll} \therefore\ \alpha = \cos^{-1}\left(\frac{3}{4}\right) & \vec{T}_1 = 48\hat{\imath} + 64\hat{\jmath} \\ \beta = \cos^{-1}\left(\frac{4}{5}\right) & \vec{T}_2 = -48\hat{\imath} + 36\hat{\jmath} \end{array}}$$

Review Problems 3: Sol'ns

(3) $P(1,2,0)$, $Q(1,3,-1)$.

$$\vec{r}(t) = \langle 1,2,0\rangle(1-t) + \langle 1,3,-1\rangle(t)$$
$$= \boxed{\langle 1, 2+t, -t\rangle}$$

(4) For line of intersection, solve $2x+y-z=0$ ①

$x+y+2z=0$ ②.

①-②: $x-3z=0 \Rightarrow x=3z$, $y=-x-2z$

Letting $z=t$, $\vec{r}(t) = \langle 3t, -5t, t\rangle = t\langle 3,-5,1\rangle$.

Angle between line and x-axis is angle θ between $\langle 3,-5,1\rangle$ and $\langle 1,0,0\rangle$: $\langle 1,0,0\rangle \cdot \langle 3,-5,1\rangle = |\langle 1,0,0\rangle|\,|\langle 3,-5,1\rangle| \cos(\theta)$

$$\Rightarrow 3 = (1)\sqrt{35}\cos(\theta)$$
$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{\sqrt{35}}\right) \approx \boxed{59.5^\circ}$$

(5) $$f(x) = \begin{cases} \dfrac{\sin(x-y)}{|x|+|y|} & \text{if } |x|+|y| \neq 0, \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

f is not continuous at $(0,0)$:

- Letting $(x,y)\to(0,0)$ along positive x-axis,
$$\frac{\sin(x-y)}{|x|+|y|} = \frac{\sin(x)}{x} \to 1.$$
- Letting $(x,y)\to(0,0)$ along line $y=x$,
$$\frac{\sin(x-y)}{|x|+|y|} = \frac{\sin(0)}{2|x|} = 0 \to 0.$$

Review Problems 3: Sol'ns

(6) Curve of intersection of $x^2+2y+2z=4$ and $y=1$ is $x^2+2(1)+2z=4$

$\Rightarrow \quad x^2+2z=2$

Letting $x=t$, curve has parametrization

$x(t)=t, \quad z(t)=1-\frac{1}{2}t^2, \quad y(t)=1.$

The point $(1,1,\frac{1}{2})$ corresponds to $t=1$ at which

$x'(1)=1, \quad y'(1)=0, \quad z'(1)=-1.$

$\therefore$ tangent vector to curve of intersection at $(1,1,\frac{1}{2})$ is $\langle 1,0,-1\rangle$.

$\therefore$ tangent line is $\boxed{\vec{r}(t)=\langle 1,1,\tfrac{1}{2}\rangle+t\langle 1,0,-1\rangle.}$

---

(7) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ intersects the coordinate axes at $x=a$, $y=b$, $z=c$, respectively.

The volume cut off from the first octant is that of the tretrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ & $(0,0,c)$, which is $\frac{1}{6}abc$.

Since $(2,1,2)$ is on the plane, the goal is to

$$\left.\begin{array}{l}\text{minimize } f(a,b,c)=\dfrac{abc}{6}\\ \text{subject to } \dfrac{2}{a}+\dfrac{1}{b}+\dfrac{2}{c}=1\end{array}\right\}.$$

By method of Lagrange Multipliers using $g(a,b,c)=\frac{2}{a}+\frac{1}{b}+\frac{2}{c}$:

$$\left.\begin{array}{l}\nabla f=\lambda\nabla g\\ g(a,b,c)=1\end{array}\right\}\Rightarrow\left.\begin{array}{ll}① & \dfrac{bc}{6}=-\lambda\dfrac{2}{a^2}\\ ② & \dfrac{ac}{6}=-\lambda\dfrac{1}{b^2}\\ ③ & \dfrac{ab}{6}=-\lambda\dfrac{2}{c^2}\\ ④ & \dfrac{2}{a}+\dfrac{1}{b}+\dfrac{2}{c}=1\end{array}\right\}\quad (*)$$

$\longrightarrow$

Review Problems 3: sol'ns

A minimum exists and at the minimum $abc \neq 0$ since $(2,1,2)$ is on the plane.

$\therefore \quad a \neq 0, \quad b \neq 0, \quad c \neq 0 \quad \& \quad \lambda \neq 0.$

$① \div ② \Rightarrow \dfrac{b}{a} = \dfrac{2b^2}{a^2} \Rightarrow a = 2b$

$② \div ③ \Rightarrow \dfrac{c}{b} = \dfrac{c^3}{2b^2} \Rightarrow c = 2b$

$\therefore ④ \Rightarrow \dfrac{\cancel{2}}{\cancel{2}b} + \dfrac{1}{b} + \dfrac{\cancel{2}}{\cancel{2}b} = 1 \Rightarrow \dfrac{3}{b} = 1 \Rightarrow b = 3$

$\therefore a = (2)(3) = 6, \quad c = (2)(3) = 6.$

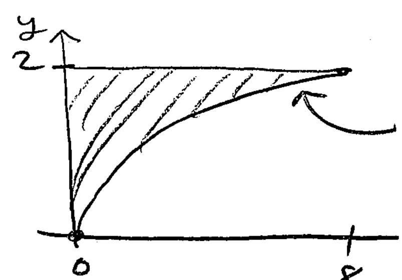
There is certainly at least one solution to the problem, and at each solution (*) must be satisfied. Since (*) has only the one solution $(a,b,c) = (6,3,6)$ then this solution must correspond to the absolute minimum of f.

$\therefore$ The plane is $\dfrac{x}{6} + \dfrac{y}{3} + \dfrac{z}{6} = 1.$

---

(8) $I = \displaystyle\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{1+y^4}\, dy\, dx$

Here D is



$y = \sqrt[3]{x} \Leftrightarrow x = y^3.$

$\therefore I = \displaystyle\int_{y=0}^{2} \int_{x=0}^{y^3} \frac{1}{1+y^4}\, dx\, dy$

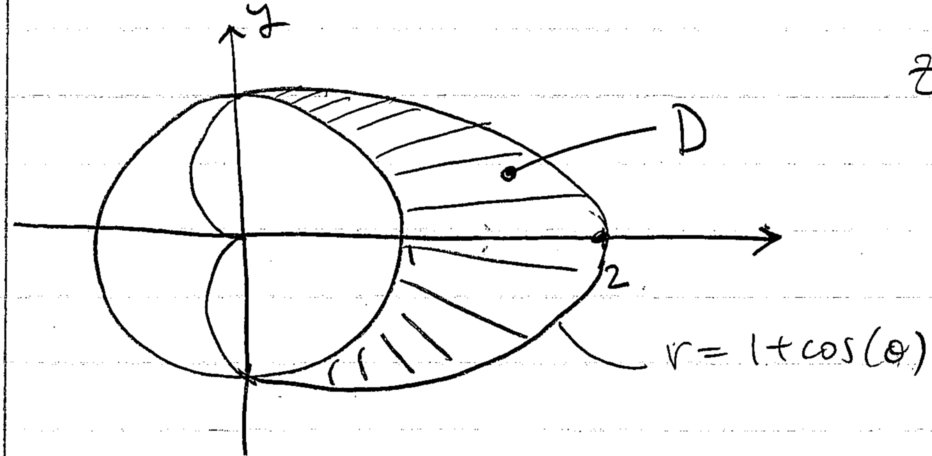
$= \displaystyle\int_0^2 \frac{y^3}{1+y^4}\, dy$

$= \left[\frac{1}{4} \ln|1+y^4|\right]_0^2$

$= \boxed{\frac{1}{4} \ln(17)}$

Review Problems 3: Soln's

(9)



$z = 7 - x$

$$V = \iiint_E 1 \, dV$$

$$= \iint_D \left(\int_{z=0}^{7-x} 1 \, dz \right) dA = \boxed{\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} \int_{z=0}^{7-r\cos\theta} 1 \, r \, dz \, dr \, d\theta}$$

Answer!

$$= \iint_D (7 - x) \, dA$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} (7 - r\cos\theta) \, r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{7r^2}{2} - \frac{r^3}{3} \cos\theta \right) \Big|_1^{1+\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{7(1+\cos\theta)^2}{2} - \frac{(1+\cos\theta)^3 \cos\theta}{3} - \frac{7}{2} + \frac{1}{3} \cos\theta \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cancel{\frac{7}{2}} + 7\cos\theta + \frac{7}{2}\cos^2\theta - \cancel{\frac{\cos\theta}{3}} - \cos^2\theta - \cos^3\theta - \frac{\cos^4\theta}{3} - \cancel{\frac{7}{2}} + \cancel{\frac{1}{3}\cos\theta} \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 7\cos\theta + \frac{5}{2}\cos^2\theta - \cos^3\theta - \frac{\cos^4\theta}{3} \, d\theta$$

$$= \frac{38}{3} + \frac{9\pi}{8}$$

Wolfram alpha!