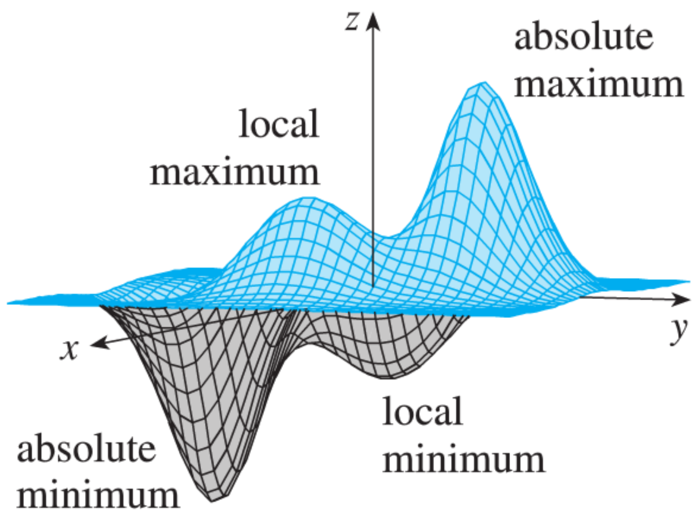
11.7: Maximum and Minimum Values:

Goal: Use partial derivatives to locate maximum and minimum values of functions of two variables:



Definition:

- f has a **local maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for every (x, y) in some disk with centre (a, b).
- f has a **local minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for every (x, y) in some disk with centre (a, b).
- f has an **absolute maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for every (x, y) in the domain of f.
- f has an **absolute minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for every (x, y) in the domain of f.

Note: the term *relative minimum* (resp. *maximum*) is equivalent to *local minimum* (resp. *maximum*). The term *global minimum* (resp. *maximum*) is equivalent to *absolute minimum* (resp. *maximum*).

Theorem: If f has a local maximum or minimum at (a, b) and both $f_x(a, b)$, $f_y(a, b)$ exist, then $f_x(a, b) = f_y(a, b) = 0$.

Proof: (in the case of local maximum) If f has a local maximum at (a, b) then $f(x, b)$ has a local maximum at $x = a$ as a function of one variable, so either $f_x(a, b) = 0$ or $f_x(a, b)$ does not exist. Since $f_x(a, b)$ exists by hypothesis it must be that $f_x(a, b) = 0$. By a similar argument $f_y(a, b) = 0$.

Definition: A point (a, b) is a **critical point** of f if $f_x(a, b) = f_y(a, b) = 0$ of if at least one of $f_x(a, b)$, $f_y(a, b)$ fails to exist.

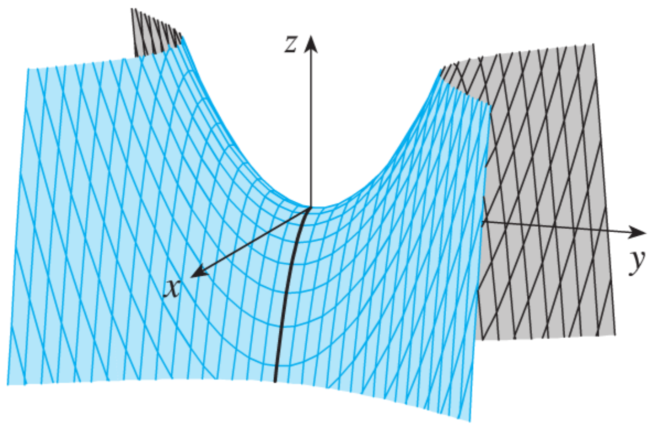
Conclusion: Local extrema occur at critical points, but not every critical point corresponds to a local extremum.

As in single variable calculus, the nature of critical points can be determined in part using

Theorem (Second Derivative Test): Suppose that f_{xx}, f_{yy}, f_{xy} and f_{yx} are all continuous on a disk with centre (a, b), and that $f_x(a, b) = f_y(a, b) = 0$. Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
- If $D < 0$ then $f(a, b)$ is neither a local minimum nor maximum; (a, b) is a *saddle point* (the graph of f crosses its tangent plane at (a, b)):



- If $D = 0$ then the test fails.

Note: D above is sometimes called the **Hessian** and the matrix $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ the *Hessian matrix*.

In the case of finding absolute extrema we again have a theorem which resembles its single variable counterpart:

Theorem (Extreme Value Theorem for Functions of Two Variables): If f is continuous on a closed and bounded set D in $\mathbb{R}^2$ then f attains an absolute maximum and an absolute minimum at some points in D.

Absolute extrema occur either at critical points (where they also qualify as relative extrema), or on the boundary of D. So to identify absolute extrema of a continuous f on a closed and bounded set D proceed as follows:

1. Find values of f at the critical points of f inside D.
2. Find the extreme values of f on the boundary of D.
3. Select the largest and smallest values of f from steps 1 and 2 above. These are, respectively, the absolute maximum and absolute minimum values of f on D.