Question 1: Use the method of Lagrange multipliers to find the absolute maximum of $f(x, y, z) = x + 3y - z$ on the sphere $x^2 + y^2 + z^2 = 4$.

Maximize $f(x,y,z) = x+3y-z$

Subject to $g(x,y,z) = x^2+y^2+z^2=4$

$$\left.\begin{array}{l}\nabla f = \lambda \nabla g \\ g(x,y,z) = 4\end{array}\right\} \Rightarrow \left\{\begin{array}{ll} 1 = 2\lambda x & (1) \\ 3 = 2\lambda y & (2) \\ -1 = 2\lambda z & (3) \\ x^2+y^2+z^2=4 & (4)\end{array}\right.$$

note: none of x, y, z or λ can be zero.

(1) & (2) $\Rightarrow \frac{1}{x} = \frac{3}{y} \Rightarrow y = 3x$

(2) & (3) $\Rightarrow \frac{3}{y} = \frac{-1}{z} \Rightarrow z = \frac{-1}{3}y = -x$

(4) $\Rightarrow x^2 + (3x)^2 + (-x)^2 = 4$

$\Rightarrow x^2+9x^2+x^2=4$

$\Rightarrow 11x^2 = 4$

$\Rightarrow$

$x = \frac{2}{\sqrt{11}}$ | $x = \frac{-2}{\sqrt{11}}$

$\therefore y = 3x = \frac{6}{\sqrt{11}}$ | $y = 3x = \frac{-6}{\sqrt{11}}$

$\therefore z = -x = \frac{-2}{\sqrt{11}}$ | $z = -x = \frac{2}{\sqrt{11}}$.

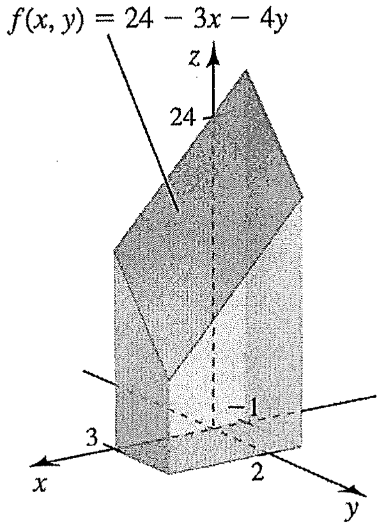
At $\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$: $f\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right) = \frac{22}{\sqrt{11}}$ } max.

At $\left(\frac{-2}{\sqrt{11}}, \frac{-6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$: $f\left(\frac{-2}{\sqrt{11}}, \frac{-6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right) = \frac{-22}{\sqrt{11}}$

$\boxed{\therefore f \text{ has an absolute maximum of } \frac{22}{\sqrt{11}} = 2\sqrt{11}}$

[10]

Question 2: Determine the volume of the following solid:



$$V = \int_{x=-1}^{3} \int_{y=0}^{2} (24-3x-4y)\,dy\,dx$$

$$= \int_{x=-1}^{3} \left[24y - 3xy - \frac{4}{2}y^2\right]_{y=0}^{2} dx$$

$$= \int_{x=-1}^{3} (48-6x-8) - (0)\,dx$$

$$= \left[40x - 3x^2\right]_{-1}^{3}$$

$$= (120-27) - (-40-3)$$

$$= \boxed{136}$$

[5]

Question 3: The average value of the function $f(x, y)$ over the region D is defined to be

$$f_{\text{ave}} = \frac{1}{A(D)} \iint_D f(x, y)\,dA$$

where $A(D)$ is the area of D. Determine the average value of $f(x, y) = (y+1)e^{x(y+1)}$ over the rectangle $R = [0, 1] \times [-1, 1]$.

$A(R) = (1)(2) = 2.$

$$\therefore f_{\text{ave}} = \frac{1}{2} \int_{y=-1}^{1} \int_{x=0}^{1} (y+1)e^{x(y+1)}\,dx\,dy$$

$$= \frac{1}{2} \int_{y=-1}^{1} \left[\cancel{(y+1)}\frac{e^{x(y+1)}}{\cancel{(y+1)}}\right]_{x=0}^{1} dy$$

$$= \frac{1}{2} \int_{y=-1}^{1} (e^{y+1} - 1)\,dy$$

$$= \frac{1}{2}\left[e^{y+1} - y\right]_{-1}^{1}$$

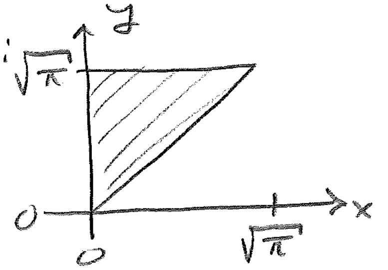
$$= \frac{1}{2}\left[(e^2-1) - (1+1)\right]$$

$$= \boxed{\frac{e^2-3}{2}}$$

[5]

Question 4: Evaluate $\displaystyle\iint_D y^2\cos(xy)\,dA$ where D is the triangular region in the first quadrant bounded by $y = x$, $y = \sqrt{\pi}$ and $x = 0$.

D:



$$\iint_D y^2\cos(xy)\,dA$$

$$= \int_{x=0}^{\sqrt{\pi}}\int_{y=x}^{\sqrt{\pi}} y^2\cos(xy)\,dy\,dx \quad \Big\}\text{ by parts; reverse order of integration}$$

$$= \int_{y=0}^{\sqrt{\pi}}\int_{x=0}^{y} y^2\cos(xy)\,dx\,dy \quad \Big\}\text{ easier}$$

$$= \int_{y=0}^{\sqrt{\pi}} \cancel{y^2}\frac{\sin(xy)}{\cancel{y}}\Big|_0^y\,dy$$

$u = y^2$

$du = 2y\,dy$

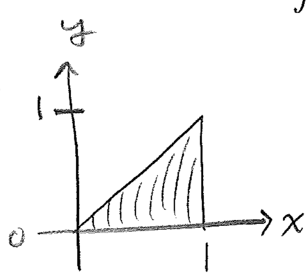
$$= \int_{y=0}^{\sqrt{\pi}} \sin(y^2)\,y\,dy$$

$$= -\frac{\cos(y^2)}{2}\Big|_0^{\sqrt{\pi}} = \boxed{1}$$

[5]

Question 5: Evaluate $\displaystyle\int_0^1\int_y^1 e^{(x^2)}\,dx\,dy$ by reversing the order of integration.

D:



$$\int_0^1\int_y^1 e^{x^2}\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} e^{x^2}\,dy\,dx$$

$$= \int_{x=0}^{1} e^{x^2}[y]_0^x\,dx$$

$u = x^2$

$du = 2x\,dx$

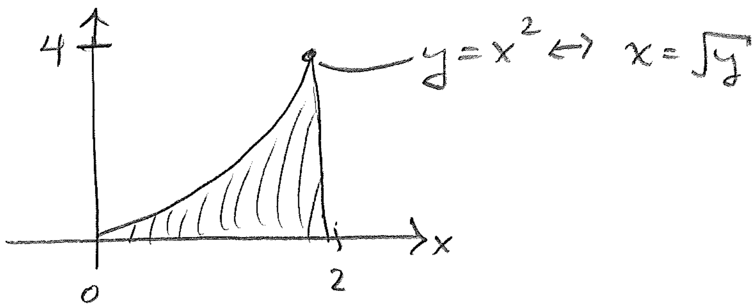
$$= \int_{x=0}^{1} e^{x^2}\cdot x\,dx$$

$$= \left[\frac{e^{x^2}}{2}\right]_0^1 = \boxed{\frac{e-1}{2}}$$

[5]

Question 6: Express the following integral with the order of integration reversed: $\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx$

D:



$$\int_{x=0}^{2} \int_{y=0}^{x^2} f(x,y)\,dy\,dx = \boxed{\int_{y=0}^{4} \int_{x=\sqrt{y}}^{2} f(x,y)\,dx\,dy}.$$

[5]

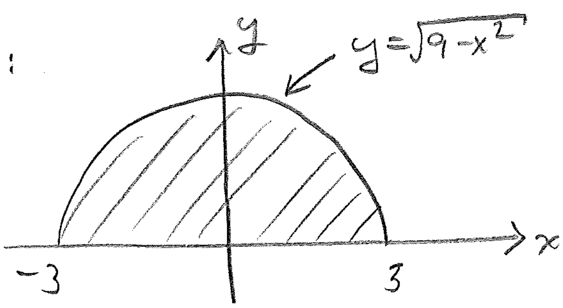
Question 7: Evaluate $\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$. (Hint: consider polar.)

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{3} \sin(r^2)\, r\,dr\,d\theta$$

$u=r^2$
$du = 2r\,dr$

$$= \int_{\theta=0}^{\pi} \left[-\frac{\cos(r^2)}{2}\right]_0^3 d\theta$$

D:



$x^2+y^2=r^2$

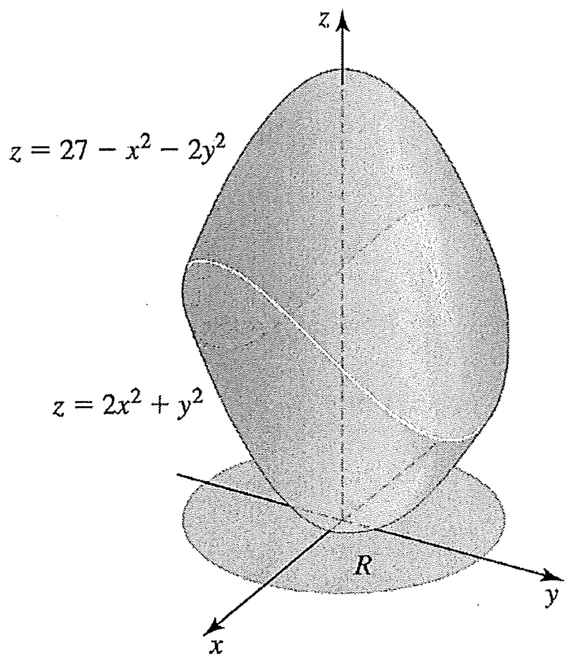
$$= \int_{\theta=0}^{\pi} \left(\frac{1-\cos(9)}{2}\right) d\theta$$

$$= \left(\frac{1-\cos(9)}{2}\right) [\theta]_0^{\pi}$$

$$= \boxed{\frac{\pi}{2}(1-\cos(9))}$$

[5]

Question 8: Determine the volume of the following solid:



$$V = \iint_R (27 - x^2 - 2y^2) - (2x^2 + y^2) \, dA$$

$$= \iint_R 27 - 3(x^2 + y^2) \, dA$$

Here R has boundary given by

$$27 - x^2 - 2y^2 = 2x^2 + y^2$$

$$3x^2 + 3y^2 = 27$$

$$x^2 + y^2 = 3^2,$$

a disk of radius 3.

Convert to polar

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} (27 - 3r^2) \, r \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{27r^2}{2} - \frac{3r^4}{4} \right]_0^3 d\theta$$

$$= \left(\frac{27(9)}{2} - \frac{3(81)}{4} \right) \left[\theta \right]_0^{2\pi}$$

$$= \left(\frac{243}{\cancel{4}_2} \right) (\cancel{2}\pi)$$

$$= \boxed{\frac{243\pi}{2}}$$

[5]