Question 1: Find an equation of the tangent plane to $z = x^2 + \ln(x - 2y)$ at the point where $x = 3$ and $y = 1$.

Let $f(x,y) = x^2 + \ln(x-2y)$.

$f(3,1) = 3^2 + \ln(3-2(1)) = 9$

$f_x(3,1) = 2x + \frac{1}{x-2y}\Big|_{(3,1)} = (2)(3) + \frac{1}{3-2(1)} = 7$

$f_y(3,1) = \frac{-2}{x-2y}\Big|_{(3,1)} = \frac{-2}{3-2(1)} = -2$

$\therefore$ Equation is $z = f(3,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1)$

$z = 9 + 7(x-3) - 2(y-1)$

or $\boxed{7x - 2y - z = 10}$

[5]

Question 2: Use a linear approximation to $f(x,y) = y + \sin(x/y)$ at the point $(0,3)$ to approximate $f(-0.1, 3.2)$.

$f(0,3) = 3 + \sin\left(\frac{0}{3}\right) = 3.$

$f_x(0,3) = \frac{1}{y}\cos\left(\frac{x}{y}\right)\Big|_{(0,3)} = \frac{1}{3}\cos\left(\frac{0}{3}\right) = \frac{1}{3}$

$f_y(0,3) = 1 - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)\Big|_{(0,3)} = 1 - \frac{0}{3^2}\cos\left(\frac{0}{3}\right) = 1$

$L(x,y) = f(0,3) + f_x(0,3)(x-0) + f_y(0,3)(y-3)$

$= 3 + \frac{1}{3}x + (1)(y-3)$

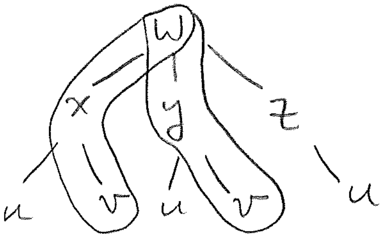
$= \frac{1}{3}x + y$

$\therefore f(-0.1, 3.2) \approx L(-0.1, 3.2) = \left(\frac{1}{3}\right)\left(\frac{-1}{10}\right) + 3.2$

$= \boxed{\frac{95}{30} \doteq 3.17}$

[5]

Question 3: Let $w = x + 2y + z^2$ where $x = u/v$, $y = u^2 + \ln(v)$ and $z = 2u$. Compute $\dfrac{\partial w}{\partial v}$ at $(u, v) = (-1, 1)$.



$$\frac{\partial w}{\partial v} = w_x x_v + w_y y_v = (1)\left(\frac{-u}{v^2}\right) + (2)\left(\frac{1}{v}\right)$$

$$\therefore \left.\frac{\partial w}{\partial v}\right|_{(u,v)=(-1,1)} = \frac{-(-1)}{1^2} + (2)\left(\frac{1}{1}\right) = \boxed{3}$$

[5]

Question 4: An ellipse has parametrization $x = 2\sqrt{2}\cos(t)$, $y = \sqrt{2}\sin(t)$ where $0 \leq t \leq 2\pi$. Let $T(x, y) = xy - 2$ be the temperature at point (x, y) on the ellipse. Determine the maximum and minimum temperatures on the ellipse.

(Hint: what does $\dfrac{dT}{dt}$ equal at a maximum or minimum?)

- At the extrema $\dfrac{dT}{dt} = 0$:
- $\dfrac{dT}{dt} = T_x \dfrac{dx}{dt} + T_y \dfrac{dy}{dt}$

$$= (y)\left(-2\sqrt{2}\sin(t)\right) + (x)\left(\sqrt{2}\cos(t)\right)$$

$$= \left(\sqrt{2}\sin(t)\right)\left(-2\sqrt{2}\sin(t)\right) + \left(2\sqrt{2}\cos(t)\right)\left(\sqrt{2}\cos(t)\right)$$

$$= 4\left[\cos^2(t) - \sin^2(t)\right]$$

- $\dfrac{dT}{dt} = 0 \Rightarrow \cos^2(t) = \sin^2(t)$

$$\Rightarrow \tan^2(t) = 1$$

$$\Rightarrow \tan(t) = \pm 1 \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

- $T = xy - 2$

$$= 2\sqrt{2}\cos(t) \cdot \sqrt{2}\sin(t) - 2$$

$$= 4\sin(t)\cos(t) - 2.$$

- $T(\pi/4) = 0$, $T(\frac{3\pi}{4}) = -4$, $T(\frac{5\pi}{4}) = 0$, $T(\frac{7\pi}{4}) = -4$.
- End points: $T(0) = T(2\pi) = -2$

$\therefore$ Maximum temperature is 0°, minimum is -4°

[5]

Question 5: Find the directional derivative of $f(x, y) = e^{xy}\sin(y)$ at $(0, \pi/4)$ in the direction of $\mathbf{v} = \langle -6, 8\rangle$.

Unit direction is $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle -6,8\rangle}{10} = \langle \frac{-3}{5}, \frac{4}{5}\rangle$.

$$D_{\vec{u}} f(0, \tfrac{\pi}{4}) = \nabla f(0, \pi/4) \cdot \vec{u}$$

$$= \langle ye^{xy}\sin(y),\ xe^{xy}\sin(y) + e^{xy}\cos(y)\rangle\Big|_{(0,\pi/4)} \cdot \vec{u}$$

$$= \langle \tfrac{\pi}{4} e^{0}\sin(\tfrac{\pi}{4}),\ 0 + e^{0}\cos(\pi/4)\rangle \cdot \vec{u}$$

$$= \langle \frac{\pi}{4\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle \cdot \langle \frac{-3}{5}, \frac{4}{5}\rangle$$

$$= \frac{-3\pi}{20\sqrt{2}} + \frac{4}{5\sqrt{2}} = \boxed{\frac{16-3\pi}{20\sqrt{2}}}$$

[5]

Question 6: Find all points (x, y) at which the direction of fastest change of $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\langle 1, 1\rangle$.

Want all (x,y) at which

$\nabla f(x,y) = k\langle 1,1\rangle$ for some constant k

$\Rightarrow \langle 2x-2, 2y-4\rangle = k\langle 1,1\rangle$

$\Rightarrow 2x-2 = k$ and $2y-4 = k$

$\therefore 2x-2 = 2y-4$

$\therefore 2y-2x = 2$

or $y-x = 1$

$\therefore$ Condition will be satisfied at all points of the line $y-x=1$

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Question 7: Find the equations of the tangent planes to the surface

$$\sin(xyz) = x + 2y + 3z$$

at all points at which $x = 2$ and $z = 0$.

At $x=2$ and $z=0$: $\sin((2)(y)(0)) = 2+2y+(3)(0)$

$\therefore\ y=-1$

Let $F(x,y,z) = \sin(xyz) - x - 2y - 3z$.

$$\nabla F(2,-1,0) = \left\langle yz\cos(xyz)-1,\ xz\cos(xyz)-2,\ xy\cos(xyz)-3 \right\rangle \Big|_{(2,-1,0)}$$

$$= \langle -1,-2,-5 \rangle.$$

So tangent plane has equation

$\nabla F(2,-1,0) \cdot \langle x-2,\ y+1,\ z-0 \rangle = 0$

$\langle -1,-2,-5 \rangle \cdot \langle x-2,\ y+1,\ z \rangle = 0$

$-(x-2)-2(y+1)-5(z)=0$

$\boxed{x+2y+5z=0}$

[5]

Question 8: Find the minimum distance from the cone $z = \sqrt{x^2+y^2}$ to the point $(-6, 4, 0)$. Explain why the solution you found does indeed correspond to the minimum.

Minimize $f(x,y,z) = \sqrt{(x+6)^2+(y-4)^2+z^2}$

Subject to $z = \sqrt{x^2+y^2}$.

$\therefore\ f(x,y,z) = g(x,y) = \sqrt{(x+6)^2+(y-4)^2+x^2+y^2}$

$$g_x = 0 \Rightarrow \frac{2(x+6)+2x}{2\sqrt{(x+6)^2+(y-4)^2+x^2+y^2}} = 0 \Rightarrow 2x+6=0 \Rightarrow x=-3.$$

$$g_y = 0 \Rightarrow \frac{2(y-4)+2y}{2\sqrt{(x+6)^2+(y-4)^2+x^2+y^2}} = 0 \Rightarrow 2y-4=0 \Rightarrow y=2.$$

$\therefore$ The minimum distance is $g(-3,2) = \sqrt{(-3+6)^2+(2-4)^2+(-3)^2+2^2} = \boxed{\sqrt{26}}$

This must be the minimum since a minimum exists, it must occur at a C.P., and there is but one C.P.

[5]

Question 9: Find all critical points of $f(x, y) = 4xy - x^4 - y^4$ and classify each as either a local maximum, local minimum or a saddle point. Carefully calculate all required derivatives and keep your work organized.

$f_x = 4y - 4x^3$; $f_{xx} = -12x^2$; $f_{xy} = 4$

$f_y = 4x - 4y^3$; $f_{yy} = -12y^2$

- $f_x = f_y = 0$?

$f_x = 0 \Rightarrow 4y - 4x^3 = 0 \Rightarrow y = x^3$

$f_y = 0 \Rightarrow 4x - 4y^3 = 0 \Rightarrow x - y^3 = 0 \Rightarrow x - (x^3)^3 = 0$

$\Rightarrow x - x^9 = 0$

$\Rightarrow x(1 - x^8) = 0$

$\Rightarrow x = 0,\ x = 1,\ x = -1$

At $x = 0$, $y = x^3 = 0$: $(0,0)$ is a CP.

$x = 1$, $y = x^3 = 1$: $(1,1)$ is a CP

$x = -1$, $y = x^3 = -1$: $(-1,-1)$ is a CP.

- f_x or f_y not exist? No such (x,y).

CP	$D = f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	Conclusion
$(0,0)$	$-16 < 0$	—	saddle point
$(1,1)$	$128 > 0$	$-12 < 0$	local max.
$(-1,-1)$	$128 > 0$	$-12 < 0$	local max.

[10]