Question 1: Determine if the space curve with parametric equations $x = t\cos(t)$, $y = t\sin(t)$, $z = t$ lies on the surface $z^2 = x^2 + y^2$.

For $x = t\cos(t)$, $y = t\sin(t)$, $z = t$,
does $x^2+y^2 = z^2$ for every t?

$$\begin{aligned} x^2+y^2 &= [t\cos(t)]^2 + [t\sin(t)]^2 \\ &= t^2\cos^2(t) + t^2\sin^2(t) \\ &= t^2\underbrace{[\cos^2(t)+\sin^2(t)]}_{1} \\ &= t^2 \\ &= z^2 \end{aligned}$$

$\boxed{\text{So } \underline{\text{yes}}\text{, the space curve does lie on } z^2 = x^2+y^2.}$

[5]

Question 2: Determine the point where the tangent line to $\mathbf{r}(t) = \langle 1+2\sqrt{t}, t^3 - t, t^3 + t\rangle$ at $(3, 0, 2)$ intersects the xy-plane.

$\vec{r}(t) = \langle 1+2\sqrt{t}, t^3-t, t^3+t \rangle = \langle 3,0,2 \rangle$ at $t=1$.

Direction vector of tangent line is

$$\vec{r}\,'(1) = \left\langle \frac{1}{\sqrt{t}}, 3t^2-1, 3t^2+1 \right\rangle \Big|_{t=1} = \langle 1, 2, 4 \rangle.$$

So tangent line is $\vec{l}(t) = \langle 3,0,2 \rangle + t\langle 1,2,4 \rangle = \langle 3+t, 2t, \underline{2+4t} \rangle$.
At point where $\vec{l}(t)$ intersects the xy-plane we have $2+4t=0$, so $t = -\frac{1}{2}$.

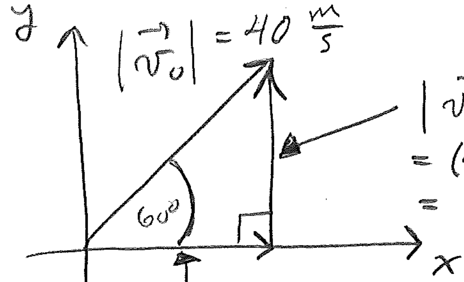
$\therefore$ point is $\vec{l}\left(-\frac{1}{2}\right) = \left\langle 3+\left(\frac{-1}{2}\right), 2\left(\frac{-1}{2}\right), 2+4\left(\frac{-1}{2}\right) \right\rangle = \boxed{\left\langle \frac{5}{2}, -1, 0 \right\rangle}$

[5]

Question 3: A projectile of mass 2 kg is launched from the origin in the direction of the positive x-axis with an initial speed of 40 m/s at an angle of 60° to the ground. The wind applies a constant horizontal opposing force of 1 N. The resulting equation of motion governing the projectile's flight is

$$2\mathbf{a}(t) = -\mathbf{i} - 2g\mathbf{j}$$

where $g = 9.8$ m/s^2 is acceleration due to gravity. How far from the launch site does the projectile land?



$|\vec{v}_0| \sin(60)\, \hat{j} = (40)\left(\frac{\sqrt{3}}{2}\right)\hat{j} = 20\sqrt{3}\, \hat{j}$

$|\vec{v}_0| \cos(60)\, \hat{\imath} = (40)\left(\frac{1}{2}\right)\hat{\imath} = 20\hat{\imath}$

$\therefore \vec{v}(0) = \vec{v}_0 = 20\hat{\imath} + 20\sqrt{3}\hat{j}$.

$$\vec{a}(t) = -\frac{1}{2}\hat{\imath} - g\hat{j}, \quad \vec{v}(0) = \vec{v}_0, \quad \vec{s}(0) = \langle 0,0 \rangle.$$

$$\therefore \vec{v}(t) = -\frac{1}{2}t\hat{\imath} - gt\hat{j} + \vec{c}_1$$

$$\vec{v}(0) = \vec{c}_1 = \vec{v}_0 \implies \vec{c}_1 = \vec{v}_0$$

$$\therefore \vec{v}(t) = \left(20 - \frac{1}{2}t\right)\hat{\imath} + \left(20\sqrt{3} - gt\right)\hat{j}$$

$$\therefore \vec{s}(t) = \left(20t - \frac{1}{2}\frac{t^2}{2}\right)\hat{\imath} + \left(20\sqrt{3}\,t - g\frac{t^2}{2}\right)\hat{j} + \vec{C}_2$$

$$\vec{s}(0) = \langle 0,0 \rangle \text{ so } \vec{C}_2 = \langle 0,0 \rangle.$$

$$\therefore \vec{s}(t) = \left(20t - \frac{t^2}{4}\right)\hat{\imath} + \left(20\sqrt{3}\,t - g\frac{t^2}{2}\right)\hat{j}$$

Projectile hits ground when $20\sqrt{3}\,t - g\frac{t^2}{2} = 0$

$$\Rightarrow t\left(20\sqrt{3} - \frac{g}{2}t\right) = 0$$

$$\Rightarrow \cancel{t=0}, \quad t = \frac{40\sqrt{3}}{g}.$$

Distance from launch site is then $\left|\vec{s}\left(\frac{40\sqrt{3}}{g}\right)\right| = 20\left(\frac{40\sqrt{3}}{g}\right) - \frac{1}{4}\left(\frac{40\sqrt{3}}{g}\right)^2 = \boxed{128.9 \text{ m}}$ **[10]**

Question 4: Neatly draw a contour map for $f(x, y) = ye^x$ showing contours corresponding to $k = -1, 0, 1$. Label the three contours.

$k=-1$: $ye^x = -1$

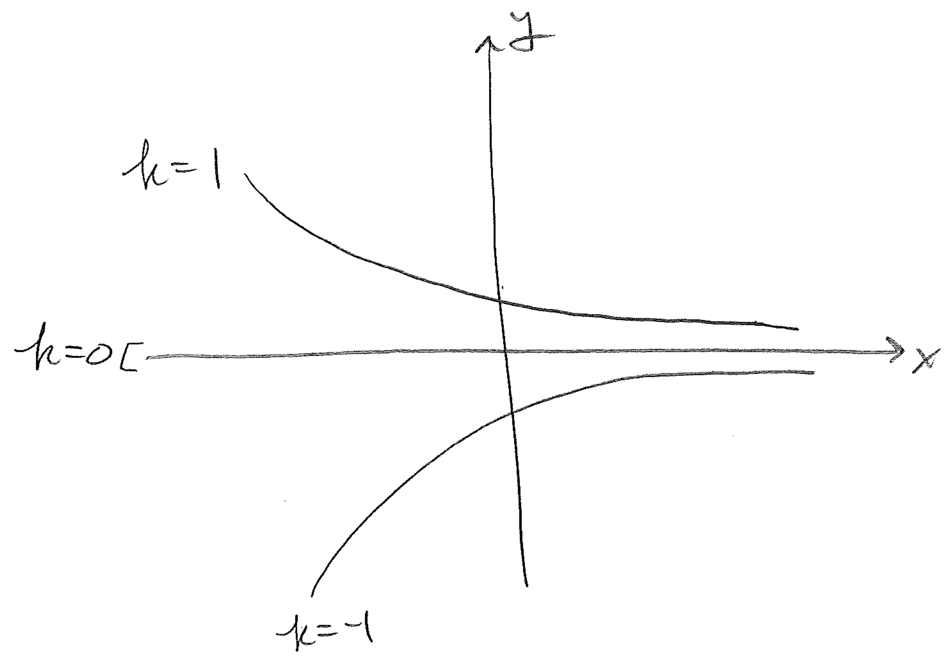
$\Rightarrow y = -e^{-x}$

$k=0$: $ye^x = 0$

$\Rightarrow y = 0$

$k=1$: $ye^x = 1$

$y = e^{-x}$



[5]

Question 5: The following limit exists; find it: $\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x^2y - x^3}$

$$= \lim_{(x,y)\to(1,1)} \frac{(x-y)(x+y)}{x^2(y-x)}$$

$$= \lim_{(x,y)\to(1,1)} \frac{-\cancel{(y-x)}(x+y)}{x^2\cancel{(y-x)}}$$

$$= \frac{-(1+1)}{1^2}$$

$$= \boxed{-2}$$

[5]

Question 6: Show that the following limit does not exist (a bit trickier): $\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4+4y^2}$

• Let $(x,y)\to(0,0)$ along x-axis, so $y=0$:

$$\frac{x^2ye^y}{x^4+4y^2} = \frac{0}{x^4+4\cdot 0^2} = 0,$$

so $\frac{x^2ye^y}{x^4+4y^2} \to 0$ as $(x,y)\to(0,0)$ along the x-axis.

• Let $(x,y)\to(0,0)$ along curve $y=x^2$:

$$\frac{x^2ye^y}{x^4+4y^2} = \frac{x^2x^2e^{x^2}}{x^4+4x^4} = \frac{e^{x^2}}{5} \to \frac{e^0}{5} = \frac{1}{5}$$

as $(x,y)\to(0,0)$ along $y=x^2$.

Since limiting values of $\frac{x^2ye^y}{x^4+4y^2}$ differ along different approach paths, the limit does not exist. **[5]**

Question 7: Let $g(x,y) = \frac{x+2y}{x^2+y^2}$. Compute $g_x(1,2) - g_y(1,2)$.

$$g_x(1,2) = \frac{\partial}{\partial x}\left[\frac{x+2y}{x^2+y^2}\right]\Bigg|_{(1,2)} = \left[\frac{(x^2+y^2)(1)-(x+2y)(2x)}{(x^2+y^2)^2}\right]\Bigg|_{(1,2)}$$

$$= \frac{(1^2+2^2)(1)-(1+(2)(2))(2)(1)}{(1^2+2^2)^2}$$

$$= -\frac{1}{5}$$

$$g_y(1,2) = \frac{\partial}{\partial y}\left[\frac{x+2y}{x^2+y^2}\right]\Bigg|_{(1,2)} = \left[\frac{(x^2+y^2)(2)-(x+2y)(2y)}{(x^2+y^2)^2}\right]\Bigg|_{(1,2)}$$

$$= \left[\frac{(1^2+2^2)(2)-(1+4)(4)}{(1^2+2^2)^2}\right] = -\frac{2}{5}$$

$$\therefore g_x(1,2) - g_y(1,2) = -\frac{1}{5} - \left(\frac{-2}{5}\right) = \boxed{\frac{1}{5}}$$

[5]

Question 8: Let $f(x, y, z) = x\sin(y^2 - z^2)$. Compute $\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial z^2}$.

$f_x = \sin(y^2 - z^2)$

$f_{xx} = 0$

$f_z = x\cos(y^2 - z^2)(-2z) = -2xz\cos(y^2 - z^2)$

$f_{zz} = -2x\cos(y^2 - z^2) + 2xz\sin(y^2 - z^2)(-2z)$

$\therefore$

$f_{xx} - f_{zz} = \boxed{+2x\cos(y^2 - z^2) + 4xz^2\sin(y^2 - z^2)}$

[5]

Question 9: Let $f(x, y, z) = xy^2z^3 + \arcsin(x\sqrt{z})$. Find f_{xzy}. (You may assume that Clairaut's Theorem applies.)

$f_{xzy} = f_{yxz}$

$f_y = 2xyz^3 + 0$

$\therefore f_{yx} = 2yz^3$

$\therefore f_{yxz} = \boxed{6yz^2}$

[5]