Question 1: Find an equation for the set of points (x, y, z) that are equidistant from the point $P(0, 2, 0)$ and the xz-plane. Simplify your equation as much as possible.

Need $\sqrt{(x-0)^2 + (y-2)^2 + (z-0)^2} = |y|$

$$\Rightarrow \quad x^2 + (y-2)^2 + z^2 = y^2$$

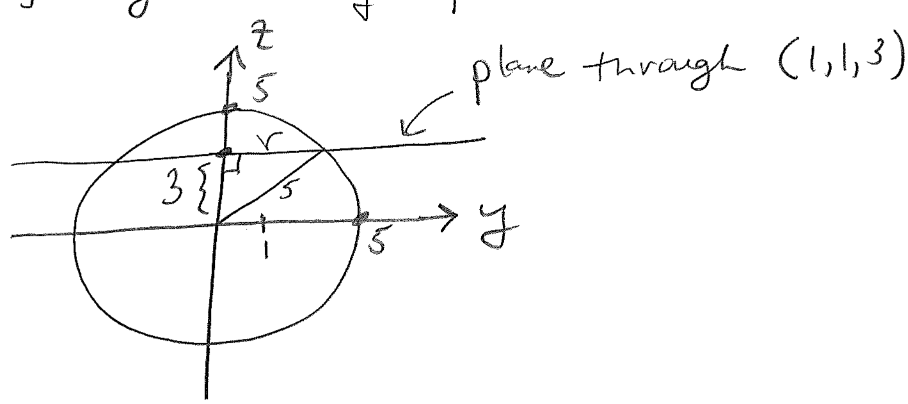
$$\Rightarrow \quad x^2 + \cancel{y^2} - 4y + 4 + z^2 = \cancel{y^2}$$

$$\Rightarrow \quad \boxed{y = \frac{x^2 + z^2 + 4}{4}}$$

[5]

Question 2: The horizontal plane through the point $P(1, 1, 3)$ slices through the sphere of radius 5 centred at the origin. Determine the radius of the slice.

Projecting onto yz-plane we have:



$$\therefore \quad r = \sqrt{5^2 - 3^2} = \boxed{4}$$

[5]

Question 3: For this question use the vectors

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

(a) Find a vector **v** of magnitude 7 in the direction of **c** .

$$\vec{v} = 7 \frac{\vec{c}}{|\vec{c}|} = 7 \frac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|} = \boxed{\left\langle \frac{7}{\sqrt{3}}, \frac{7}{\sqrt{3}}, \frac{7}{\sqrt{3}} \right\rangle}$$

[2]

(b) Determine the angle between **a** and **b** .

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\therefore \theta = \cos^{-1}\left[\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right]$$

$$= \cos^{-1}\left[\frac{\langle 2,-1,3\rangle \cdot \langle 3,-2,1\rangle}{|\langle 2,-1,3\rangle| \, |\langle 3,-2,1\rangle|}\right]$$

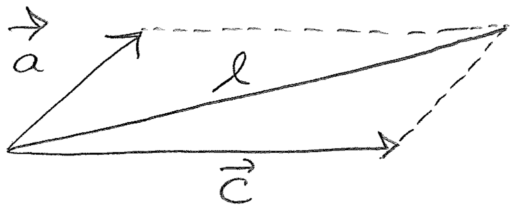
$$= \cos^{-1}\left[\frac{6+2+3}{\sqrt{14}\sqrt{14}}\right]$$

$$= \cos^{-1}\left[\frac{11}{14}\right]$$

$$\approx \boxed{38.2^\circ}$$

[2]

(c) If **a** and **c** are placed with their tails at the origin they define two sides of a parallelogram. Determine the length of the longest diagonal of that parallelogram.



length $l = |\vec{a} + \vec{c}|$

$$= |\langle 2,-1,3\rangle + \langle 1,1,1\rangle|$$

$$= |\langle 3,0,4\rangle|$$

$$= \boxed{5}$$

[3]

(d) Are **a**, **b** and **c** coplanar?

Is $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$?

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & -1 & 3 \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (2)(-3) - (-1)(2) + (3)(5)$$

$$= -6 + 2 + 15$$

$$= 11$$

$$\neq 0$$

[3]

$\therefore$ vectors are <u>not</u> coplanar.

Question 4: A plane flying due east (that is, in the **i** direction) at 200 km/hour experiences a north-easterly wind (that is, a wind blowing in the **i**+**j** direction) of 40 km/hour. Determine the resulting <u>speed</u> of the plane.

$$\vec{v}_{plane} = 200\,\hat{\imath} = \langle 200, 0\rangle$$

$$\vec{v}_{wind} = 40\,\frac{\hat{\imath}+\hat{\jmath}}{|\hat{\imath}+\hat{\jmath}|} = 40\,\frac{\langle 1,1\rangle}{\sqrt{2}} = \left\langle \frac{40}{\sqrt{2}}, \frac{40}{\sqrt{2}} \right\rangle$$

$\therefore$ Speed of the plane is $|\vec{v}_{plane} + \vec{v}_{wind}|$

$$= \left|\langle 200, 0\rangle + \left\langle \frac{40}{\sqrt{2}}, \frac{40}{\sqrt{2}}\right\rangle\right|$$

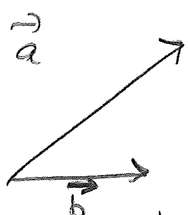
$$= \left|\left\langle 200 + \frac{40}{\sqrt{2}}, \frac{40}{\sqrt{2}}\right\rangle\right|$$

$$= \sqrt{\left(200 + \frac{40}{\sqrt{2}}\right)^2 + \left(\frac{40}{\sqrt{2}}\right)^2}$$

$$\approx \boxed{230\ \frac{km}{hr}}$$

[5]

Question 5: Let **a** = ⟨4, −1, 0⟩ and **b** = ⟨0, 1, 1⟩. Find vectors **u** and **v** such that **u** ∥ **b**, **v** ⊥ **b**, and **a** = **u**+**v** .



$\vec{u} = \text{proj}_{\vec{b}}\vec{a}$

$\vec{v} = \vec{a} - \vec{u}$

$$\vec{u} = \text{proj}_{\vec{b}}\vec{a} = \left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \left(\langle 4,-1,0\rangle \cdot \frac{\langle 0,1,1\rangle}{|\langle 0,1,1\rangle|}\right)\left(\frac{\langle 0,1,1\rangle}{|\langle 0,1,1\rangle|}\right)$$

$$= \left(\frac{-1}{\sqrt{2}}\right)\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$$= \boxed{\left\langle 0, \frac{-1}{2}, \frac{-1}{2}\right\rangle}$$

$$\vec{v} = \vec{a} - \vec{u} = \langle 4,-1,0\rangle - \left\langle 0, \frac{-1}{2}, \frac{-1}{2}\right\rangle$$

$$= \boxed{\left\langle 4, \frac{-1}{2}, \frac{1}{2}\right\rangle}$$

[5]

Question 6: Calculate $(\mathbf{a} \times \mathbf{b}) \cdot (2\mathbf{a} + 3\mathbf{b})$.

$$= 2\underbrace{(\vec{a}\times\vec{b})\cdot\vec{a}}_{=0} + 3\underbrace{(\vec{a}\times\vec{b})\cdot\vec{b}}_{=0}$$

$$= \boxed{0}$$

[3]

Question 7: If the angle between unit vectors **u** and **v** is $\pi/6$, determine $|\mathbf{u} + \mathbf{v}|$.

$$|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$\text{So } |\vec{u}+\vec{v}| = \sqrt{\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} + 2\vec{u}\cdot\vec{v}}$$

$$= \sqrt{\cancel{|\vec{u}|^2}^{\,1} + \cancel{|\vec{v}|^2}^{\,1} + 2\cancel{|\vec{u}|}^{\,1}\cancel{|\vec{v}|}^{\,1}\cancel{\cos\left(\tfrac{\pi}{6}\right)}^{\,\frac{\sqrt{3}}{2}}}$$

$$= \sqrt{1+1+2\left(\tfrac{\sqrt{3}}{2}\right)}$$

$$= \boxed{\sqrt{2+\sqrt{3}}}$$

[3]

Question 8: Find an equation of the line through $(1, -1, 1)$ that is parallel to the line

$$x + 2 = \frac{y}{2} = z - 3$$

You may state the answer using any form of a line you like.

Need direction vector of $\{\ \}$ ∴ transform to parametric form:

$$\left.\begin{array}{l} x+2 = t \Rightarrow x = t-2 \\ \frac{y}{2} = t \Rightarrow y = 2t \\ z-3 = t \Rightarrow z = 3+t \end{array}\right\} \therefore \langle x,y,z\rangle = \langle t-2, 2t, 3+t\rangle$$

$$= \langle -2, 0, 3\rangle + t\underbrace{\langle 1,2,1\rangle}_{\text{direction.}}$$

∴ Line through $(1,-1,1)$ in direction $\langle 1,2,1\rangle$ is

$$\boxed{\vec{r} = \langle 1,-1,1\rangle + t\langle 1,2,1\rangle}$$

[4]

Question 9: Find an equation of the plane containing the points $P(1,1,1)$, $Q(1,2,-1)$ and $R(0,1,0)$.

normal to plane : $\vec{n} = \vec{PQ} \times \vec{PR}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix}$$

$$= \langle -1, 2, 1 \rangle$$

Using $R(0,1,0)$ and $\vec{n} = \langle -1,2,1 \rangle$:

$$(\langle x,y,z \rangle - \langle 0,1,0 \rangle) \cdot \langle -1,2,1 \rangle = 0$$

$$-x + 2(y-1) + z = 0$$

$$\boxed{-x + 2y + z = 2}$$

[5]

Question 10: Find an equation of the plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $\underbrace{5x + 2y + z = 1}$.

↳ normal to plane is $\vec{n} = \langle 5, 2, 1 \rangle$.

For a point on plane, let $t = 0$ in $\left.\begin{array}{l} x = 1+t \\ y = 2-t \\ z = 4-3t \end{array}\right\} (1,2,4)$

$\therefore$ Plane is $(\langle x,y,z \rangle - \langle 1,2,4 \rangle) \cdot \langle 5,2,1 \rangle = 0$

$$5(x-1) + 2(y-2) + z - 4 = 0$$

$$\boxed{5x + 2y + z = 13}$$

[5]
