Strategy for Optimization Problems

Optimization problems are those in which we seek the absolute maximum or minimum value of some function, usually in the context of an applied problem. The problems are often of a physical nature. As with related rate problems, it is important to use a systematic approach to set up and solve optimization problems. Here are some guidelines, but keep in mind that every problem is different:

1. Thoroughly read and understand the problem. Be clear about what quantity (function) is being maximized or minimized, and the variables on which it depends.

2. Sketch a diagram, if applicable, introduce variables and summarize the given information.

3. State the objective function (the function being optimized), noting constraints, if any, on the variables.

   For example, if a problem asks you to find the maximum possible product of two non-negative numbers whose sum of squares is 16, the problem can be stated

   Maximize $P = xy$ subject to $x^2 + y^2 = 16$

   Here P is the objective function while the constraint on x and y is $x^2 + y^2 = 16$.

4. If necessary, reduce the objective function to one variable using constraints or other relationships between variables. For example, in the problem of finding the maximum product of two non-negative numbers whose sum of squares is 16, from the constraint $x^2 + y^2 = 16$ we could write $y = \sqrt{16 - x^2}$, so that the product function $P = xy$ can be reduced to

   $$P(x) = x\sqrt{16 - x^2}$$

5. Now restate the objective, this time noting the feasible domain:

   Maximize $P(x) = x\sqrt{16 - x^2}$ where $0 \leq x \leq 4$.

6. Now apply the methods we've seen previously for identifying extrema. Once the extreme value has been found, be sure to justify using the closed interval method, the first derivative test, or the second derivative test that it does indeed correspond to the desired optimum value of the objective function.

7. Write a clear conclusion, being careful to give a complete answer to the original given question. Some optimization questions may ask for the extreme value of the objective function, while others may ask for the variable values that produce the extreme value.

Example 1

A box with square base and open top is to have a volume of 32 m^3. Find the dimensions of the box that minimize the amount of material used in its construction.

Example 2

A person is in a boat 1 km from a straight shoreline. From the nearest point on shore, the person's beach house is located 3 km down the beach. The person can row the boat 1 km/hr and walk 3 km/hr. To which point on the shoreline should the person row in order to get to the beach house in the shortest amount of time?

Example 3

A cone is inscribed in a larger cone of height and base radius both equal to 3 m, as shown. Find the maximum volume of the smaller cone.