1. **[15 points]** Evaluate the following limits, if they exist. If a limit does not exist state this as well as a reason why it does not exist. If a limit is an infinite limit, specify ∞ or $-\infty$ as appropriate. You may use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

(a) $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$ $= \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$

$= \lim_{x\to 3} \frac{\cancel{(x+1-4)}}{\cancel{(x-3)}(\sqrt{x+1}+2)}$

$= \boxed{\frac{1}{4}}$

(b) $\lim_{x\to 0} \frac{\tan(4x)}{x \sec x}$ $= \lim_{x\to 0} \frac{\sin(4x)}{\cos(4x)} \cdot \frac{1}{x} \cdot \cos x$

$= \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{\cos(x)}{\cos(4x)} \cdot 4$

$= \boxed{4}$

(c) $\lim_{x\to 4^-} \frac{x^2-3x+2}{x-4}$ $\begin{matrix} \} \to 6 \\ \} \to 0^- \end{matrix}$

$= \boxed{-\infty}$

(d) $\lim_{x\to -\infty} \frac{e^x + e^{-4x}}{2+3e^{-4x}}$ $\begin{matrix} \div e^{-4x} \\ \div e^{-4x} \end{matrix}$

$= \lim_{x\to -\infty} \frac{e^{5x}+1}{2e^{4x}+3}$

$= \boxed{\frac{1}{3}}$

(e) $\lim_{x\to\infty} \sqrt{25x^2+10x} - 5x$ $= \lim_{x\to\infty} \frac{(\sqrt{25x^2+10x}-5x)}{1} \cdot \frac{\sqrt{25x^2+10x}+5x}{\sqrt{25x^2+10x}+5x}$

$= \lim_{x\to\infty} \frac{\cancel{25x^2}+10x\cancel{-25x^2}}{\sqrt{25x^2+10x}+5x}$

$= \lim_{x\to\infty} \frac{10\cancel{x}}{\cancel{x}\sqrt{25+\frac{10}{x}}+5\cancel{x}}$

$= \boxed{1}$

2. **[5 points]** Consider the function $f(x)$ defined below:

$$f(x) = \begin{cases} \dfrac{3x^2+4x+1}{x^2-x-2} & \text{for } x < -1 \\ x + c & \text{for } x \geq -1 \end{cases}$$

where c is a constant. Find the value of c that makes $f(x)$ continuous on $(-\infty, \infty)$.

If $x \neq -1$, f is defined by a rational function with non-zero denominator, so is continuous.

At $x=-1$, we must have

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$$

$$\lim_{x\to -1^-} \frac{3x^2+4x+1}{x^2-x-2} = \lim_{x\to -1^+} x+c = (-1)+c$$

$$\Rightarrow \lim_{x\to -1^-} \frac{(3x+1)\cancel{(x+1)}}{(x-2)\cancel{(x+1)}} = -1+c = -1+c$$

$$\Rightarrow \frac{-2}{-3} = -1+c$$

$$\therefore\ c = 1+\frac{2}{3} = \boxed{\frac{5}{3}}$$

3. **[5 points]** Use the limit definition of the derivative to find $f'(x)$ for $f(x) = \dfrac{x}{x+1}$. (Note: a score of 0 will be given if $f'(x)$ is found using the differentiation rules.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h}{x+h+1} - \frac{x}{x+1}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{x^2}+\cancel{hx}+\cancel{x}+h-\cancel{x^2}-\cancel{hx}-\cancel{x}}{(x+h+1)(x+1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}}{(x+h+1)(x+1)}\right]$$

$$= \boxed{\frac{1}{(x+1)^2}}$$

4. **[15 points]** Differentiate the following functions. You do not need to simplify your answers.

(a) $y = e^{-7x}\ln x$

$$y' = -7e^{-7x}\ln x + e^{-7x}\frac{1}{x}$$

(b) $y = 5\tan(3^x)$

$$y' = 5\sec^2(3^x)\cdot 3^x \cdot \ln 3$$

(c) $y = \dfrac{\sqrt[4]{x^3}}{\cos(4x) - 3x^5} = \dfrac{x^{\frac{3}{4}}}{\cos(4x) - 3x^5}$

$$y' = \frac{[\cos(4x) - 3x^5]\frac{3}{4}x^{-\frac{1}{4}} - x^{\frac{3}{4}}[-\sin(4x)\cdot 4 - 15x^4]}{(\cos(4x) - 3x^5)^2}$$

(d) $y = \sqrt{x}\log_5(x^5 + x^2) = x^{\frac{1}{2}}\log_5(x^5 + x^2)$

$$y' = \frac{1}{2}x^{-\frac{1}{2}}\log_5(x^5 + x^2) + x^{\frac{1}{2}}\frac{1}{(x^5 + x^2)\ln 5}\cdot(5x^4 + 2x)$$

(e) $y = \csc^4(e^x + x)$

$$y' = 4\csc^3(e^x + x)\cdot[-\csc(e^x + x)\cot(e^x + x)]\cdot(e^x + 1).$$

5. [10 points]

(a) Use implicit differentiation to find $\frac{dy}{dx}$ for $\sin(xy^2) = e^{2y}$.

$$\frac{d}{dx}\left[\sin(xy^2)\right] = \frac{d}{dx}\left[e^{2y}\right]$$

$$\cos(xy^2)\cdot\left[y^2 + x\cdot 2yy'\right] = e^{2y}\cdot 2y'$$

$$y^2\cos(xy^2) + 2xy\cos(xy^2)y' = 2e^{2y}y'$$

$$\left[2xy\cos(xy^2) - 2e^{2y}\right]y' = -y^2\cos(xy^2)$$

$$\therefore y' = \frac{y^2\cos(xy^2)}{2e^{2y} - 2xy\cos(xy^2)}$$

(b) Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = \left(\frac{1+x^2}{7^x}\right)^{\sin x}$.

$$\ln y = \ln\left[\left(\frac{1+x^2}{7^x}\right)^{\sin x}\right]$$

$$\ln y = (\sin x)\left[\ln(1+x^2) - \ln(7^x)\right]$$

$$\ln y = \sin x\left[\ln(1+x^2) - x\ln(7)\right]$$

$$\frac{1}{y}\cdot y' = \cos x\left[\ln(1+x^2) - x\ln(7)\right] + \sin x\left[\frac{2x}{1+x^2} - \ln(7)\right]$$

$$\therefore y' = \left(\frac{1+x^2}{7^x}\right)^{\sin x}\left[\cos x\left(\ln(1+x^2) - x\ln(7)\right) + \sin x\left(\frac{2x}{1+x^2} - \ln(7)\right)\right]$$

6. **[8 points]** Suppose the position of an object moving horizontally after t seconds, $0 \le t \le 7$, is given by $s(t) = 2t^3 - 15t^2 + 24t$ where t is measured in seconds and distance is measured in meters.

(a) Find the velocity and acceleration of the object at time t.

$$v(t) = s'(t) = 6t^2 - 30t + 24 \ \frac{m}{s}$$

$$a(t) = v'(t) = s''(t) = 12t - 30 \ \frac{m}{s^2}$$

(b) Find the interval(s) when the object is moving to the right and the interval(s) when the object is moving to the left.

Object is moving to right when $s'(t) > 0$, moving to left when $s'(t) < 0$:

$$s'(t) = 0 \Rightarrow 6[t^2 - 5t + 4] = 0$$
$$\Rightarrow 6(t-4)(t-1) = 0$$
$$\Rightarrow t = 1, \ t = 4$$

0 1 4 7

[1/2 2 5]

$s'(t)$: + 0 − 0 +

$\therefore$ Object is moving to right for $0 < t < 1$ and $4 < t < 7$, moving to left for $1 < t < 4$.

(c) Find the acceleration of the object when its velocity is 60 m/s.

$$v(t) = s'(t) = 60 \Rightarrow 6t^2 - 30t + 24 = 60$$
$$\Rightarrow t^2 - 5t + 4 = 10$$
$$\Rightarrow t^2 - 5t - 6 = 0$$
$$\Rightarrow (t-6)(t+1) = 0$$
$$\Rightarrow t = 6, \ \cancel{t = -1}$$

$$\therefore a(6) = s''(6) = 12(6) - 30$$
$$= \boxed{42 \ \frac{m}{s^2}}$$

7. [**12 points**] Let $g(x) = 2x^3 - 3x^2 - 12x + 20$.

(a) Find an equation of the tangent line to $g(x)$ at $x = -3$.

$g'(x) = 6x^2 - 6x - 12$

$g'(-3) = 6(-3)^2 - 6(-3) - 12 = 60$

Also, $g(-3) = -54 - 27 + 36 + 20 = -25$

$\therefore$ equation of tangent line is

$\boxed{y + 25 = 60(x+3)}$

or $\boxed{y = 60x + 155}$

(b) Find all points (x, y) on the graph of $g(x)$ where the tangent lines are parallel to the line $2y + 24x = 1$.

$2y + 24x = 1$

$y = \frac{-24x + 1}{2}$

$y = -12x + \frac{1}{2}$ } slope $m = -12$

Solve $g'(x) = -12$:

$6x^2 - 6x \cancel{-12} = \cancel{-12}$

$6x(x-1) = 0$

$x = 0$; $x = 1$

$\therefore$ $y = g(0) = 20$; $y = g(1) = 7$

$\boxed{\therefore \text{ points are } (0,20) \text{ and } (1,7).}$

(c) Find the absolute maximum and minimum values of $g(x)$ on the interval $[-2, 3]$.

$g'(x) = 0 \Rightarrow 6x^2 - 6x - 12 = 0$

$\Rightarrow 6[x^2 - x - 2] = 0$

$\Rightarrow 6(x-2)(x+1) = 0$

$\Rightarrow x = -1, \ x = 2$

x	$g(x) = 2x^3 - 3x^2 - 12x + 20$
-2	$-16 - 12 + 24 + 20 = 16$
-1	$-2 - 3 + 12 + 20 = 27$
2	$16 - 12 - 24 + 20 = 0$
3	$54 - 27 - 36 + 20 = 11$

$\therefore$ g has an abs. max. of 27 at $x = -1$; and an abs. min. of 0 at $x = 2$.

8. **[4 points]** Let $f(x) = e^x + k$. Find the value of k so that $y = 4x + 5$ is a tangent line to $f(x)$.

Suppose $y = 4x+5$ is tangent to $y = f(x)$ at $x = a$.
then $f'(a) = 4 \Rightarrow e^a = 4 \Rightarrow a = \ln(4)$.

Also, $f(a) = 4a+5 \Rightarrow e^{\ln(4)} + k = 4\ln(4) + 5$

$\Rightarrow 4 + k = 4\ln(4) + 5$

$\Rightarrow \boxed{k = 4\ln(4) + 1}$

9. **[6 points]** The values of functions $f(x)$ and $g(x)$ and their derivatives at $x = 0$ and $x = 1$ are shown in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	-3	1	2
1	3	-2	5	-4

(a) Let $h(x) = f(x)[g(x)]^2$. Find $h'(0)$.

$h'(x) = f'(x)\,[g(x)]^2 + f(x)\,2[g(x)]\cdot g'(x)$

$\Rightarrow h'(0) = f'(0)\,[g(0)]^2 + f(0)\cdot 2[g(0)]\cdot g'(0)$

$= (-3)[1]^2 + (1)(2)(1)(2)$

$= \boxed{1}$

(b) Let $h(x) = \dfrac{f(x)}{g(x)+1}$. Find $h'(1)$.

$h'(x) = \dfrac{[g(x)+1]\,f'(x) - f(x)\,g'(x)}{[g(x)+1]^2}$

$\Rightarrow h'(1) = \dfrac{[g(1)+1]\,f'(1) - f(1)\,g'(1)}{[g(1)+1]^2}$

$= \dfrac{(5+1)(-2) - (3)(-4)}{[5+1]^2}$

$= \dfrac{-12+12}{6^2} = \boxed{0}$

10. **[6 points]** Let $f(x) = \dfrac{x^2}{2} - 3x + 2\ln x$. Determine the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Observe: domain of f is $(0, \infty)$.

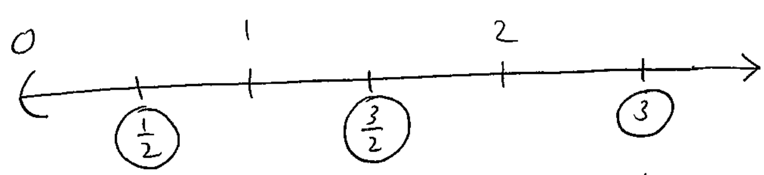
$$f'(x) = x - 3 + \frac{2}{x}$$

$$= \frac{x^2 - 3x + 2}{x}$$

$$= \frac{(x-1)(x-2)}{x}$$

$f'(x) = 0$? $x = 1$, $x = 2$.

$f'(x)$ not exist ? no such x in domain.



$f'(x)$: $+$ $-$ $+$

$f(x)$: $\nearrow$ $\searrow$ $\nearrow$

$\therefore$ f is increasing on $(0,1) \cup (2, \infty)$

f is decreasing on $(1, 2)$.

11. **[10 points]** A function and its derivatives are given below:

$$f(x) = \frac{3x}{x^2+3} \qquad f'(x) = \frac{9-3x^2}{(x^2+3)^2} \qquad f''(x) = \frac{6x^3-54x}{(x^2+3)^3}$$

(a) Find the intervals on which $f(x)$ is concave up and the intervals on which $f(x)$ is concave down.

$$f''(x) = \frac{6x(x^2-9)}{(x^2+3)^3} = \frac{6x(x-3)(x+3)}{(x^2+3)^3}$$

$f''(x)=0$? $x=0,\ 3,\ -3.$

$f''(x)$ not exist? no such x.

	-4	-3	-1	0	1	3	4
$f''(x)$:	$-$		$+$		$-$		$+$
$f(x)$:	$\frown$		$\smile$		$\frown$		$\smile$

$\boxed{\therefore f \text{ is concave up on } (-3,0)\cup(3,\infty). \quad f \text{ is concave down on } (-\infty,-3)\cup(0,3).}$

(b) Find the points of inflection of $f(x)$.

$f(-3) = \frac{-9}{12} = -\frac{3}{4}$

$f(0) = 0$

$f(3) = \frac{9}{12} = \frac{3}{4}$

$\boxed{\therefore f \text{ has inflection points } (-3,-\tfrac{3}{4}),\ (0,0),\ (3,\tfrac{3}{4}).}$

(c) Use the Second Derivative Test to find the local maximum and local minimum values of $f(x)$.

$$f'(x) = \frac{9-3x^2}{(x^2+3)^2} = \frac{3(3-x^2)}{(x^2+3)^2} = \frac{3(\sqrt{3}-x)(\sqrt{3}+x)}{(x^2+3)^2}$$

$f'(x)=0$? $x=\sqrt{3},\ x=-\sqrt{3}$

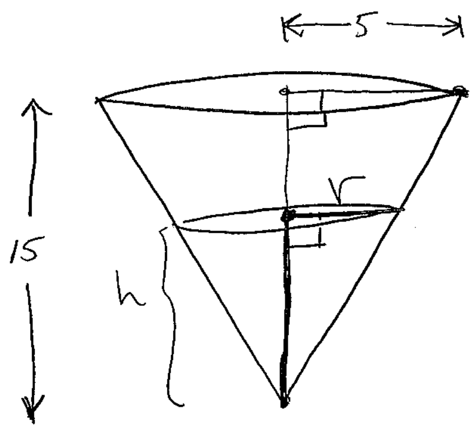
$f'(x)$ not exist? no such x.

$$f''(\sqrt{3}) = \frac{6(\sqrt{3})^3-54\sqrt{3}}{((\sqrt{3})^2+3)^3} < 0$$

$$f''(-\sqrt{3}) = \frac{6(-\sqrt{3})^3-54(-\sqrt{3})}{((-\sqrt{3})^2+3)^3} > 0$$

$\boxed{\therefore f \text{ has a rel. max. of } f(\sqrt{3}) = \tfrac{\sqrt{3}}{2} \text{ at } x=\sqrt{3}. \quad f \text{ has a rel. min. of } f(-\sqrt{3}) = -\tfrac{\sqrt{3}}{2} \text{ at } x=-\sqrt{3}.}$

12. [**9 points**] A water tank has the shape of an inverted circular cone. The height of the tank is 15 meters and the diameter of the base of cone is 10 meters. The tank is full of water. The water is pumped out of the tank at a rate of 4 m^3/min. At what rate is the height of the water changing when the water is 4 meters deep? (The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)



$\frac{dV}{dt} = -4 \frac{m^3}{min}$

Find $\frac{dh}{dt}$ when $h = 4$ m.

By similar Δ's : $\frac{r}{h} = \frac{5}{15}$

$\therefore r = \frac{1}{3} h$

$\therefore V = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h$

$V = \frac{\pi}{27} h^3$

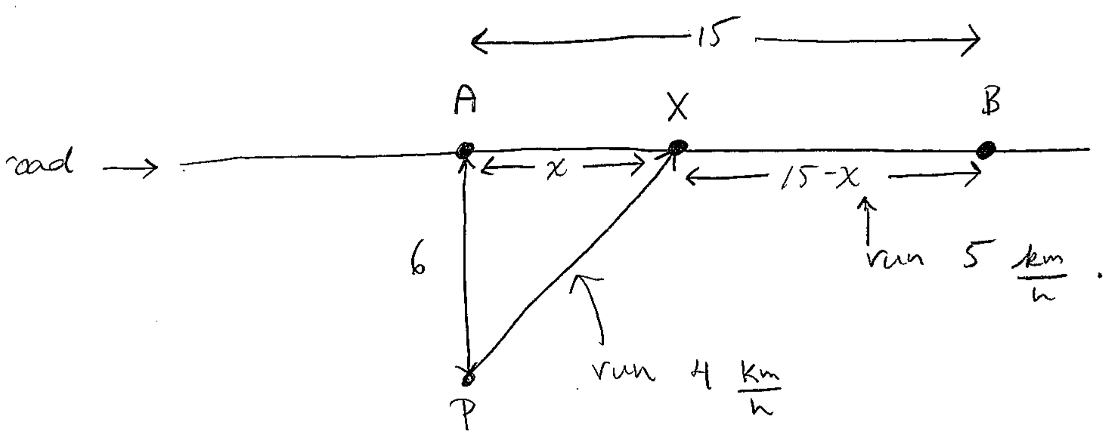
$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \frac{dh}{dt}$

When $h = 4$: $-4 = \frac{\pi}{9} (4)^2 \cdot \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{-9}{4\pi}$

$\therefore$ Water depth is decreasing at $\frac{9}{4\pi} \frac{m}{min}$.

13. **[10 points]** A jogger is running through the desert, 6 kilometers due south of the nearest point A on a straight road that runs east-west. The jogger wants to get to a point B on the road, which is 15 kilometers east of point A. The jogger can run 4 km/h in the desert and 5 km/h on the road. Find the point X on the road, between A and B, that the jogger should run to in order to minimize their travel time to point B.



Time to run from P to X : $\dfrac{\sqrt{6^2+x^2}\ \text{km}}{4\ \frac{\text{km}}{\text{h}}} = \dfrac{\sqrt{36+x^2}}{4}$ h

Time to run from X to B : $\dfrac{15-x\ \text{km}}{5\ \frac{\text{km}}{\text{h}}} = \dfrac{15-x}{5}$ h.

$\therefore$ total travel time is $T(x) = \dfrac{\sqrt{36+x^2}}{4} + \dfrac{15-x}{5}$ h

We wish to minimize $T(x)$ on $[0,15]$.

$$T'(x) = \frac{d}{dx}\left[\frac{(36+x^2)^{\frac{1}{2}}}{4} + \frac{15-x}{5}\right]$$

$$= \frac{2x}{8\sqrt{36+x^2}} - \frac{1}{5}$$

$$T'(x) = 0 \Rightarrow \frac{\cancel{2}x}{4\cancel{8}\sqrt{36+x^2}} = \frac{1}{5}$$

$$\Rightarrow 5x = 4\sqrt{36+x^2}$$

$$\Rightarrow 25x^2 = 16(36+x^2)$$

$$\Rightarrow 9x^2 = (16)(36)$$

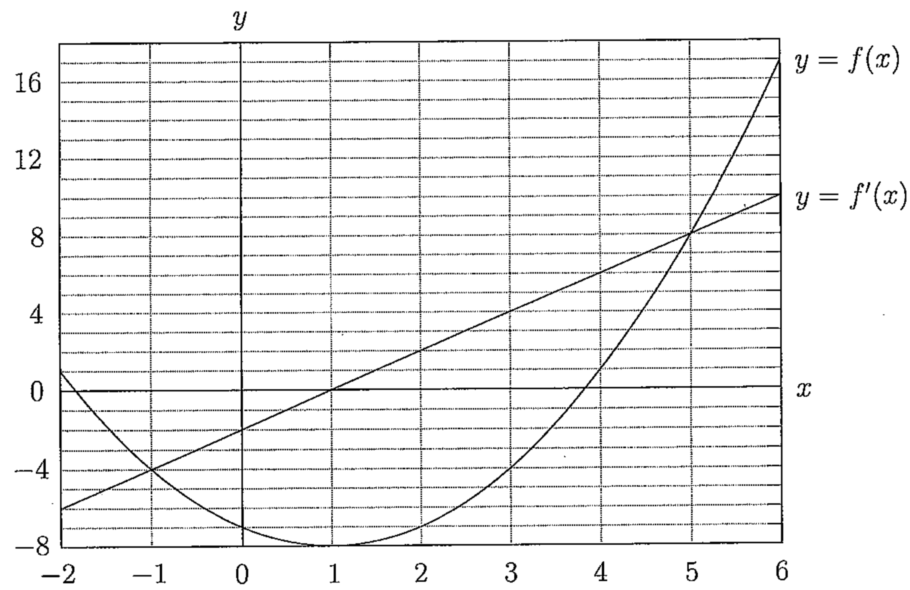
$$3x = 24$$

$$x = 8$$

Using closed interval method : $\begin{cases} T(0) = \frac{9}{2} \text{ h} \\ T(8) = \frac{5}{2} + \frac{7}{5} = \frac{39}{10} \leftarrow \text{min.} \\ T(15) = \frac{\sqrt{261}}{4} > \frac{\sqrt{256}}{4} = 4 \end{cases}$

$\therefore$ $T(x)$ is minimized at $x=8$, so X should be chosen 8 km east of A

14. **[5 points]** Suppose $g(x) = f(x)e^{-x}$ where the graphs of the function $f(x)$ and its derivative $f'(x)$ are shown in the figure below. Find the x-value(s) of the local extrema of $g(x)$. Make sure that you identify each x-value as a local maximum or a local minimum.



$$g'(x) = f'(x)e^{-x} - f(x)e^{-x}$$
$$= e^{-x}[f'(x) - f(x)]$$

$g'(x) = 0 \Rightarrow f'(x) - f(x) = 0$
$\Rightarrow f'(x) = f(x)$
$\Rightarrow x = -1, \quad x = 5.$

-2, -1, 5, 6

$-\frac{3}{2}$, 0, $\frac{11}{2}$

$g'(x) = e^{-x}[f'(x) - f(x)]$: $-$ $+$ $-$

$g(x)$: $\searrow$ $\nearrow$ $\searrow$

$\therefore$ $x = -1$ corresponds to a local minimum.
$x = 5$ corresponds to a local maximum.