Math 121 - Summary of Limit Laws

G.Pugh

Sep 17 2018

Limit Laws

Assumptions

In the following, suppose:

- c represents a constant (a fixed number)
- The limits
$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$
both exist

Sum Law

- $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- In words: *The limit of a sum is the sum of the limits*
- Example: $\lim_{x \to \pi} \left[\sqrt{x} + \sin x\right] = \left(\lim_{x \to \pi} \sqrt{x}\right) + \left(\lim_{x \to \pi} \sin x\right)$

Difference Law

- $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

- In words: *The limit of a difference is the difference of the limits*

- Example: $\lim_{x \to -3} \left[\frac{1}{x} - x^3 \right] = \left(\lim_{x \to -3} \frac{1}{x} \right) - \left(\lim_{x \to -3} x^3 \right)$

Constant Multiplier Law

- $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
- In words: *The limit of a constant times a function is the constant times the limit of the function.*
- Example: $\lim_{x \to \sqrt{2}} \left[\frac{3}{7\sqrt{x}} \right] = \frac{3}{7} \left(\lim_{x \to \sqrt{2}} \frac{1}{\sqrt{x}} \right)$

Product Law

- $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right) \left(\lim_{x \to a} g(x)\right)$
- In words: *The limit of a product is the product of the limits*
- Example:
  $\lim_{x \to 0} \left[(x^2 + 2)(1 + \cos x)\right] = \left(\lim_{x \to 0} (x^2 + 2)\right) \left(\lim_{x \to 0} (1 + \cos x)\right)$

Quotient Law

- $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$.
- In words: *The limit of a quotient is the quotient of the limits*
- Example: $\lim_{x \to 0} \left[\frac{x^2 + 2}{1 + \cos x} \right] = \frac{\lim_{x \to 0} (x^2 + 2)}{\lim_{x \to 0} (1 + \cos x)}$

Power Law

- $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer.
- In words: *The limit of a power is the power of the limit*
- Example: $\lim_{x \to \pi} [x + \tan x]^{1000} = \left[\lim_{x \to \pi} (x + \tan x)\right]^{1000}$

Root Law

- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer, and where $\lim_{x \to a} f(x) > 0$ if n is even.

- In words: *The limit of a root is the root of the limit*

- Example: $\lim_{x \to 1} \sqrt{x^2 + 5x^3} = \sqrt{\lim_{x \to 1} \left(x^2 + 5x^3\right)}$

Particular Limit Results

Constants

- $\lim_{x \to a} c = c$
- Example: $\lim_{x \to 3} \sqrt{2\pi} = \sqrt{2\pi}$

Limit of $f(x) = x$

- $\lim_{x \to a} x = a$
- Example: $\lim_{x \to 5} x = 5$

Polynomials

- Using the Sum, Difference, Constant Multiplier and Power Laws:

  If $f(x)$ is a polynomial, (for eg. $f(x) = 5x^3 - \pi x^2 - \frac{1}{2}$), then $\lim_{x \to a} f(x) = f(a)$.

- Example:
  $$\lim_{x \to -1} 5x^3 - \pi x^2 - \frac{1}{2} = 5(-1)^3 - \pi(-1)^2 - \frac{1}{2} = -\pi - \frac{11}{2}$$

Rational Functions

- Using the previous result and the Quotient Law:

  If $f(x)$ and $g(x)$ are polynomials and $g(a) \neq 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$.

- Example: $\lim_{x \to 2} \frac{2x^3 - x}{3x + 1} = \frac{2(2)^3 - (2)}{3(2) + 1} = \frac{14}{7} = 2$

Trigonometric Functions

- $\lim_{x \to a} \sin(x) = \sin(a)$
- $\lim_{x \to a} \cos(x) = \cos(a)$

- Example: $\lim_{x \to \pi/6} \sin(x) = \sin(\pi/6) = \frac{1}{2}$

Direct Substitution Property

- Putting together these limit results we have the *Direct Substitution Property*:
  - If $f(x)$ is a function defined using sums, differences, products or quotients involving polynomials, $\sin(x)$, or $\cos(x)$, and
  - if a is in the domain of $f(x)$ (that is, $f(a)$ is defined)

  then

$$\lim_{x \to a} f(x) = f(a)$$

- Example:

$$\lim_{x \to \pi} \frac{-2x^3 - \sin^2(x)}{\cos^3(x)} = \frac{-2\pi^3 - \sin^2(\pi)}{\cos^3(\pi)} = \frac{-2\pi^3 - 0}{(-1)^3} = 2\pi^3$$

Some Advice

When evaluating limits, try to apply the *Direct Substitution Property* first.

If direct substitution fails, then resort to more sophisticated techniques.