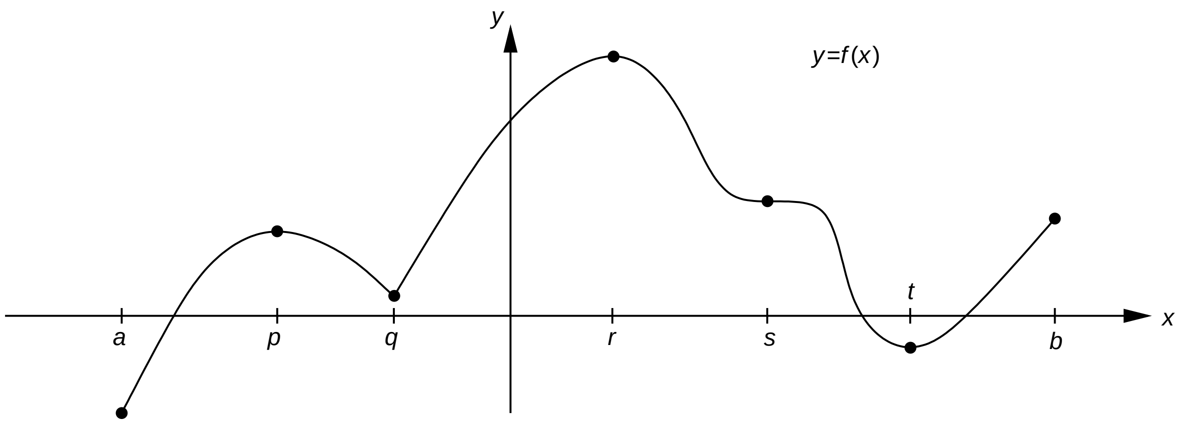
First Derivatives and Extreme Values

What information does $f'(x)$ give us about the maximum and minimum values of a function $f(x)$? Using the following graph of a general function let's introduce some terminology and make some observations:



Here f has domain $D = [a, b]$.

Recall:

- an **interval** is a continuous segment of the real line. For example, $[0, 1]$, $(-\pi, 7]$, $(0, \infty)$ and $(-\infty, \infty)$ are all intervals.
- A **closed interval** is an interval which includes its endpoints. For example, $[0, 1]$ is closed, but $(-\pi, 7]$, $(0, \infty)$ and $(-\infty, \infty)$ are not.

Definitions and Theorems

absolute (or global) maximum: f has an absolute maximum of $f(c)$ at $x = c$ if $f(c) \geq f(x)$ for every x in D.

absolute (or global) minimum: f has an absolute minimum of $f(c)$ at $x = c$ if $f(c) \leq f(x)$ for every x in D.

extreme values (or extrema) of f: the absolute maximum of f together with the absolute minimum.

relative (or local) maximum: f has a relative maximum of $f(c)$ at $x = c$ if $f(c) \geq f(x)$ for every x in an open interval containing c.

relative (or local) minimum: f has a relative minimum of $f(c)$ at $x = c$ if $f(c) \leq f(x)$ for every x in an open interval containing c.

Extreme Value Theorem: If f is continuous on $[a, b]$ then f attains an absolute maximum $f(c)$ and an absolute minium $f(d)$ for some numbers c and d in $[a, b]$.

So, referring to the graph above, we would say:

- f has an absolute maximum of $f(r)$ at $x = r$;
- f has an absolute minimum of $f(a)$ at $x = a$;
- f has relative maxima of $f(p)$ at $x = p$ and $f(r)$ at $x = r$;
- f has a relative minima of $f(q)$ at $x = q$ and $f(t)$ at $x = t$

Note:

(i) End points can correspond to absolute but not relative maxima or minima.

(ii) A point interior to the interval can correspond to both a relative and absolute maximum or minimum.

Another definition:

critical number: a critical number of a function f is a number c in the domain of f such that

(i) $f'(c) = 0$, or
(ii) $f'(c)$ does not exist

Referring to our graph, $x = p$, $x = q$, $x = r$, $x = s$ and $x = t$ are critical numbers of f. Notice the behaviour of the graph of $y = f(x)$ at each of these critical numbers. Indeed,

Fermat's Theorem: If f has a relative maximum or relative minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

Fermat's Theorem tells us that relative extrema must occur at critical numbers, however it does not say that every critical number corresponds to a relative extremum—look at $x = s$ in our graph above.

Now observe: absolute extrema must occur inside (a, b), in which case they are also relative extrema, or at the endpoints $x = a$ or $x = b$. This gives us a simple method for determining absolute extrema:

Closed Interval Method: To determine the absolute extrema of a continuous function f on a closed interval $[a, b]$:

(i) Evaluate f at the critical numbers in (a, b).

(ii) Evaluate $f(a)$ and $f(b)$.

(iii) Select the largest and smallest values from (i) and (ii) – these are the absolute maximum and minimum, respectively, of f on $[a, b]$.