Question 1: Suppose f is an invertible function with $f(4) = 5$ and that the slope of the tangent line to $y = f(x)$ at $x = 4$ is 2/3. Find the equation of the tangent line to the graph of $y = f^{-1}(x)$ at $x = 5$.

Since $f(4) = 5$, $f^{-1}(5) = 4$, so $(5,4)$ is a point on the tangent line.

Slope of tangent is $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{(\frac{2}{3})} = \dfrac{3}{2}$.

$\therefore$ Equation of tangent line is

$$\boxed{y - 4 = \frac{3}{2}(x-5)}$$

[5]

Question 2:

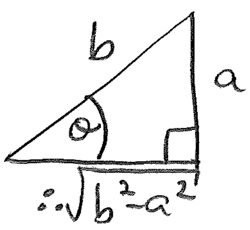
(a) Determine $\arccos(\cos(7\pi/4)) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \boxed{\frac{\pi}{4}}$

[2]

(b) Simplify $\tan(\sin^{-1}(a/b))$. Your final answer should not contain any trigonometric or inverse trigonometric functions.

Let $\theta = \sin^{-1}(\frac{a}{b})$,

So $\sin(\theta) = \frac{a}{b}$:



$\therefore \tan(\sin^{-1}(\frac{a}{b}))$

$= \tan(\theta)$

$= \boxed{\dfrac{a}{\sqrt{b^2 - a^2}}}$

[3]

Question 3:

(a) Let $f(x) = \arcsin(e^x)$. Find $f'(x)$.

$$f'(x) = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \boxed{\frac{e^x}{\sqrt{1-e^{2x}}}}$$

[2]

(b) Let $f(x) = e^{\arctan(1+x)}$. Find $f'(0)$.

$$f'(x) = e^{\arctan(1+x)} \cdot \frac{1}{1+(1+x)^2} \cdot 1$$

$$f'(0) = e^{\arctan(1)} \cdot \frac{1}{1+1^2} = \boxed{\frac{e^{\pi/4}}{2}}$$

[3]

Question 4: Find the following limits, if they exist:

(a) $\lim_{x\to 0} \frac{e^{x^2} - \cos(x)}{x^2} \sim \frac{0}{0}$

$$\overset{H}{=} \lim_{x\to 0} \frac{2xe^{x^2} + \sin(x)}{2x} \sim \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2e^{x^2} + (2x)^2 e^{x^2} + \cos(x)}{2} = \boxed{\frac{3}{2}}$$

[2]

(b) $\lim_{x\to 0^+} \sqrt{x}\ln(x) \sim 0 \cdot (-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}} \sim \text{"}\frac{-\infty}{\infty}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -2x^{1/2} = \boxed{0}$$

[3]

Question 5: Find the absolute maximum and absolute minimum values of $\underbrace{f(x) = \frac{4x}{x^2+4}}_{\text{continuous}}$ on the interval $\underbrace{[0, 5]}_{\text{closed.}}$.

$$f'(x) = \frac{(x^2+4)(4) - 4x(2x)}{(x^2+4)^2}$$

$$= \frac{4x^2+16-8x^2}{(x^2+4)^2}$$

$$= \frac{16-4x^2}{(x^2+4)^2}$$

$$= \frac{4(4-x^2)}{(x^2+4)^2}$$

outside $[0,5]$, so ignore.

- $f'(x)=0$? $x = -2, 2$
- $f'(x)$ not exist? no such x.

x	$f(x) = \frac{4x}{x^2+4}$	
0	0	← abs. min
2	1	← abs. max.
5	$\frac{20}{29}$	

∴ f has an abs. max. of 1 at $x=2$;
f has an abs. min. of 0 at $x=0$.

[10]

Question 6: For this question use the function $f(x) = 1 + 4x^3 + x^4$.

(a) Find the intervals of increase and decrease of f.

Domain $(-\infty, \infty)$.

$f'(x) = 12x^2 + 4x^3 = 4x^2(3+x)$

- $f'(x) = 0$? $x = 0, -3$
- $f'(x)$ not exist? no such x.

critical numbers :		-3		0	
test numbers :	-4		-1		1
$f'(x) = \boxed{4x^2}(3+x)$: (always ⊕)	$-$	0	$+$	0	$+$
$f(x) = 1 + 4x^3 + x^4$:	↘	-26	↗	1	↗

$\therefore$ f is decreasing on $(-\infty, -3)$ and

increasing on $\underbrace{(-3, 0) \cup (0, \infty)}_{\text{or } (-3, \infty)}$.

[8]

(b) State the ~~x-coordinates corresponding to the~~ relative (or local) extrema of f.

f has a rel. min. of -26 at $x = -3$.

[2]

Question 7: For this question again use the function $f(x) = 1 + 4x^3 + x^4$.

(a) Find the intervals of concavity of f.

$$f''(x) = \frac{d}{dx}\left[f'(x)\right] = \frac{d}{dx}\left[12x^2 + 4x^3\right] = 24x + 12x^2 = 12x(2+x)$$

- $f''(x) = 0$? $x = 0, -2$
- $f''(x)$ not exist? no such x.

doubly crit. num. :		-2		0	
test numbers :	-3		-1		1
$f''(x) = 12x(2+x)$:	$+$	0	$-$	0	$+$
$f(x) = 1 + 4x^3 + x^4$:	$\cup$	-15	$\cap$	1	$\cup$

$\therefore$ $y = f(x)$ is concave up on $(-\infty, -2) \cup (0, \infty)$,
concave down on $(-2, 0)$

[8]

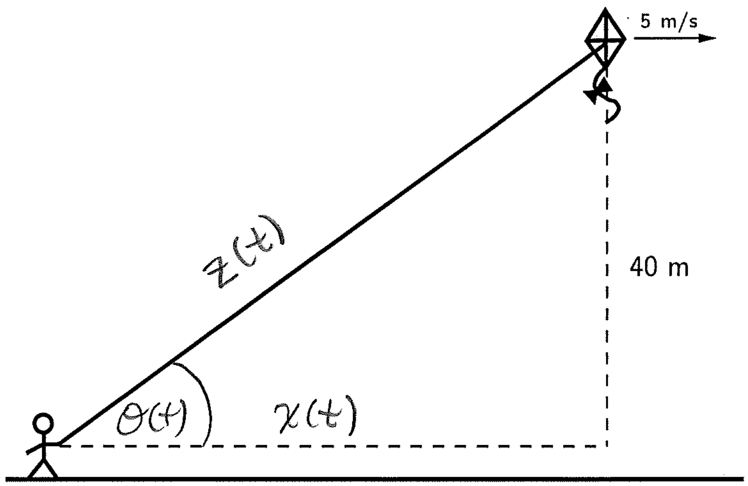
(b) State the inflection points of the graph of $y = f(x)$.

Inflection points are $(-2, -15)$ and $(0, 1)$.

[2]

BONUS:

(a) A girl is flying a kite which maintains a constant altitude of 40 m above her hand. The wind carries the kite horizontally away from her at a rate of 5 m/s. At what rate is the length of string from the girl to the kite changing when the string length is exactly 50 m?



$\frac{dx}{dt} = 5 \frac{m}{s}$.

Find $\frac{dz}{dt}$ when $z = 50$ m.

$z = \sqrt{x^2 + 40^2}$

$\frac{dz}{dt} = \frac{1}{2}(x^2+40^2)^{-\frac{1}{2}}(2x)\frac{dx}{dt}$

When $z = 50$, $x = \sqrt{50^2 - 40^2} = 30$

$\therefore \frac{dz}{dt} = \frac{1}{2}(30^2+40^2)^{-\frac{1}{2}}(2)(30)(5)$

$= \frac{1}{\cancel{(2)}\cancel{(50)}_{10}} \cancel{(2)}(30)\cancel{(5)}$

$= 3$

$\therefore$ String getting longer by $3 \frac{m}{s}$

[7]

(b) Referring to part (a), at what rate is the angle between the string and the horizontal changing at that same instant?

Find $\frac{d\theta}{dt}$ when $z = 50$.

$\tan(\theta) = \frac{40}{x}$, so

$\theta = \arctan\left(\frac{40}{x}\right)$

$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{40}{x}\right)^2} \cdot \frac{-40}{x^2}\frac{dx}{dt}$

$= \frac{-40}{x^2+40^2}\frac{dx}{dt}$

When $z = 50$, $x = 30$, so

$\frac{d\theta}{dt} = \left(\frac{-40}{30^2+40^2}\right)(5) = \frac{-40}{(50)^2} \cdot 5 = \frac{-2}{25}$

$\therefore$ θ is shrinking at $\frac{2}{25} \frac{\text{rad}}{s}$

[3]