Question 1: Find the following derivatives (it is not necessary to simplify your answers, but marks will be deducted for improper use of notation):

(a) $f(x) = 5(1 + x + x^2)^3$

$\boxed{f'(x) = 15(1+x+x^2)^2(1+2x)}$

[2]

(b) $y = \sin(\pi t^3)$

$\boxed{y' = \cos(\pi t^3)(3\pi t^2)}$

[2]

(c) $g(x) = \frac{\tan(e^x)}{7} = \frac{1}{7}\tan(e^x)$

$\boxed{g'(x) = \frac{1}{7}\sec^2(e^x)\,e^x}$

[3]

(d) $y = \sqrt{2x - \ln(x)} = (2x - \ln(x))^{1/2}$

$\boxed{y' = \frac{1}{2}(2x - \ln(x))^{-1/2}\left(2 - \frac{1}{x}\right)}$

[3]

Question 2: Find the following derivatives (it is not necessary to simplify your answers, but marks will be deducted for improper use of notation):

(a) $f(x) = \csc\left(\sqrt{1+x^2}\right) = \csc\left(\left(1+x^2\right)^{1/2}\right)$

$$\boxed{f'(x) = -\csc\left(\left(1+x^2\right)^{\frac{1}{2}}\right)\cot\left(\left(1+x^2\right)^{\frac{1}{2}}\right)\cdot\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}\cdot 2x}$$

[3]

(b) $y = \cos^2(e^{\pi t})$

$$\boxed{y' = -2\cos\left(e^{\pi t}\right)\sin\left(e^{\pi t}\right)e^{\pi t}\pi}$$

[2]

(c) $g(x) = \dfrac{(1-x^2)^5}{(1+\sin(x))^2}$

$$\boxed{g'(x) = \frac{\left(1+\sin(x)\right)^2 5\left(1-x^2\right)^4(-2x) - \left(1-x^2\right)^5 2\left(1+\sin(x)\right)\cos(x)}{\left(1+\sin(x)\right)^4}}$$

[3]

(d) $y = 2^{\sqrt{x}}\log_2(x)$

$$\boxed{y' = 2^{\sqrt{x}}\ln(2)\cdot\frac{1}{2\sqrt{x}}\log_2(x) + 2^{\sqrt{x}}\frac{1}{x\ln(2)}}$$

[2]

Question 3: Determine the equation of the tangent line to the curve

$$x^2y^2 - 2 = 2\cos(\pi y)$$

at the point $(x, y) = (1, 2)$.

$$\frac{d}{dx}\left[x^2y^2-2\right] = \frac{d}{dx}\left[2\cos(\pi y)\right]$$

$$2xy^2 + x^2\,2yy' = -2\pi\sin(\pi y)\,y'$$

at $(x,y) = (1,2)$:

$$(2)(1)(2^2) + (1^2)(2)(2)y' = \cancel{-2\pi\sin(2\pi)\,y'} \to 0$$

$$8 + 4y' = 0$$

$$y' = -\frac{8}{4} = -2$$

$\therefore$ Tangent line has equation

$$\boxed{y - 2 = -2(x-1) \quad \text{or} \quad y = -2x + 4}$$

[5]

Question 4: Use a linear approximation (or differentials) to estimate the value of $(0.9)^7$.

$f(x) = x^7$, $a = 1$

$f(a) = f(1) = 1$

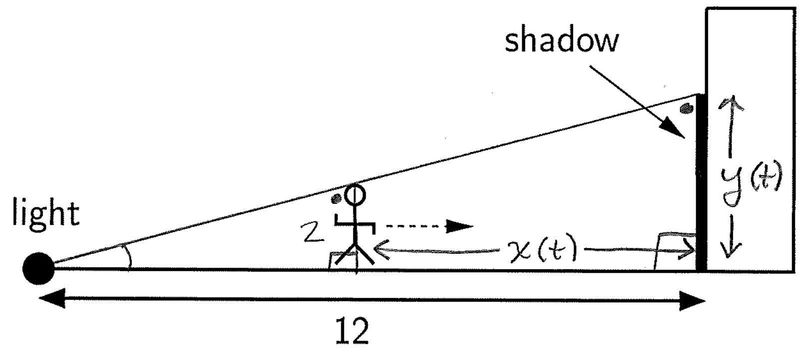
$f'(x) = 7x^6$, so $f'(a) = f'(1) = 7$

$\therefore\ L(x) = f(a) + f'(a)(x-a) = 1 + 7(x-1)$

$$\begin{aligned}\therefore\ (0.9)^7 &= f(0.9)\\ &\approx L(0.9)\\ &= 1 + 7(0.9-1)\\ &= \boxed{0.3 \text{ or } \frac{3}{10}}\end{aligned}$$

[5]

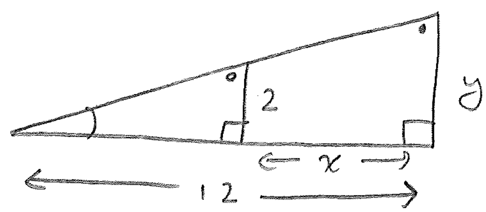
Question 5: A spotlight on the ground shines on a wall 12 m away and a man 2 m tall walks from the spotlight to the building at a speed of 1/2 m/s. How fast is the length of the man's shadow on the building decreasing when he is 6 m from the building?



$$\frac{dx}{dt} = -\frac{1}{2}\ \frac{m}{s},$$

Find $\frac{dy}{dt}$ when $x = 6$ m.

By similar triangles



$$\frac{2}{12-x} = \frac{y}{12}$$

$$\therefore\ y = \frac{24}{12-x}$$

$$\frac{dy}{dt} = \frac{-24}{(12-x)^2}\left(-\frac{dx}{dt}\right).$$

When $x = 6$, $\frac{dy}{dt} = \frac{-24}{(12-6)^2}\left(+\frac{1}{2}\right) = \frac{-1}{3}\ \frac{m}{s}$.

$\therefore$ Shadow length is decreasing by $\frac{1}{3}\ \frac{m}{s}$.

[10]

Question 6: Find y' where $y = (1+x)^{1/x}$ (logarithmic differentiation may help here.)

$$\ln(y) = \ln\left[(1+x)^{\frac{1}{x}}\right]$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\frac{1}{x}\ln(1+x)\right]$$

$$\frac{1}{y}y' = \left(\frac{-1}{x^2}\right)\ln(1+x) + \left(\frac{1}{x}\right)\left(\frac{1}{1+x}\right)$$

$$y' = (1+x)^{\frac{1}{x}}\left[\left(\frac{-1}{x^2}\right)\ln(1+x) + \frac{1}{x(1+x)}\right]$$

[5]

Question 7: The graph of the exponential function $f(x) = Ca^x$ passes through the points $(1, 6)$ and $(3, 24)$. Determine the values of the constants a and C.

$f(1) = 6$, so $\quad Ca^1 = 6 \quad ①$

$f(3) = 24$, so $\quad Ca^3 = 24 \quad ②.$

notice $C \neq 0$

$② \div ①$: $\frac{\cancel{C}a^3}{\cancel{C}a} = \frac{24}{6}$ so $a^2 = 4$,

$\therefore a = 2$ ← $a \neq -2$ since $a > 0$ for an exponential function,

Using ① now: $C \cdot 2 = 6$

so $C = \frac{6}{2} = 3$.

$\boxed{\therefore a = 2,\ C = 3,\ \text{so}\quad f(x) = 3 \cdot 2^x}$

Check: $f(1) = 3 \cdot 2^1 = 6$ ✓; $f(3) = 3 \cdot 2^3 = 24$ ✓

[5]