Elementary Row Operations

The following **Elementary Row Operations (EROs)** change the form of a matrix without changing the solution of the corresponding system of equations:

1. Interchange any two rows [notation example: $r_1 \leftrightarrow r_2$]
2. Multiply a row through by a non-zero constant [notation example: $R_3 = (-2)r_3$]
3. Add a constant of one row to another row [notation example: $R_1 = (-3)r_2 + r_1$]

Row Echelon Form

A matrix is said to be in **Row Echelon Form (REF)** if

1. The first non-zero entry in any row is a 1 (called a **leading 1**).
2. The leading 1 in any row is located to the right of the leading 1 of any row above.
3. Any rows consisting entirely of zeros are at the bottom of the matrix.

Example:

$$\begin{bmatrix} 1 & -4 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is in REF, but

$$\begin{bmatrix} 1 & -4 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is not (why?).

Reduced Row Echelon Form

A matrix is said to be in **Reduced Row Echelon Form (RREF)** if, in addition to being in REF,

4. Any column containing a leading 1 has zeros elsewhere in the column.

Example:

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF, but

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is not (why?). Nor is

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Gaussian Elimination Algorithm

To put a matrix into REF:

1. Interchange rows (if necessary) so that the first non-zero entry in the top row is located as far to the left as possible.
2. Use EROs to reduce the first entry in the top row to a leading 1. This can always be done by multiplying the top row by the reciprocal of the first entry in that row. A constant multiple of some other row can also be added to the top row to achieve this. Avoid introducing fractions if possible.
3. Now add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zero.
4. Now, **without changing or using the top row**, go to step 1 and apply the procedure to the submatrix consisting of all rows below that containing the most recently used leading 1.
5. Proceed until there are no more rows.

Gauss-Jordan Elimination Algorithm

To put a matrix into RREF, first put it into REF using Gaussian elimination procedure above, then perform an extra step:

6. Beginning with the last non-zero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

Interpreting the Outcome of the Algorithm

Each variable associated with a leading 1 in the REF (or RREF) is called a **leading variable**. Variables not associated with leading 1s are called **free variables**.

Gaussian (or Gauss-Jordan) elimination will result in exactly one of the following three outcomes:

- Some row of the REF (or RREF) will have zeros in all but the right-most position. In this case the system is **inconsistent** (has no solution.) Otherwise...
- The REF (or RREF) will have fewer non-zero rows than variables. In this case the system is **consistent** with infinitely many solutions. Solve for the leading variables in terms of the free variables which are considered **parameters** of the solution. Otherwise...
- The REF (or RREF) will have the same number of non-zero rows as variables and there is a single solution to the system. The system is **consistent** (meaning it has at least one solution).