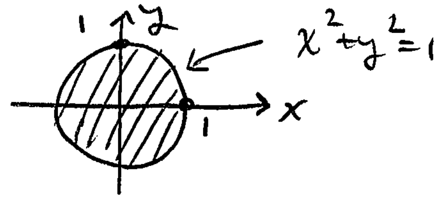
Question 1: A flat circular plate occupies the region $x^2 + y^2 \leq 1$. The temperature at point (x, y) is given by $T(x, y) = x^2 + 2y^2 - x$. Determine the hottest and coldest temperatures on the plate.

Find absolute maximum and minimum of

$T(x,y) = x^2 + 2y^2 - x$ on D:



In interior of D:

$$\left.\begin{aligned} T_x &= 2x-1 \\ T_y &= 4y \end{aligned}\right\} \quad T_x = T_y = 0 \quad \text{at} \quad \left(\tfrac{1}{2}, 0\right), \quad T\left(\tfrac{1}{2}, 0\right) = \boxed{-\tfrac{1}{4}}$$

min.

On boundary of D:

$$y^2 = 1 - x^2, \quad \text{so} \quad T = x^2 + 2(1-x^2) - x$$

$$= 2 - x - x^2, \quad -1 \leq x \leq 1$$

$$\frac{dT}{dx} = -1 - 2x = 0 \quad \text{at} \quad x = \frac{-1}{2}$$

x	$T = 2 - x - x^2$
-1	2
$-\frac{1}{2}$	$\frac{9}{4}$ max.
1	0

$\therefore$ T has an absolute maximum of $\frac{9}{4}$,

and an absolute minimum of $-\frac{1}{4}$.

[10]

Question 2: Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible. Be sure to justify that the solution you find does indeed correspond to the desired minimum.

Minimize $f(x,y,z) = x^2+y^2+z^2$

subject to $x+y+z=9$.

$\therefore z = 9-x-y \Rightarrow f(x,y,z) = g(x,y) = x^2+y^2+(9-x-y)^2$.

Let's minimize $g(x,y)$ on $\mathbb{R}^2$:

$g_x = 2x - 2(9-x-y) = 4x+2y-18$

$g_y = 2y - 2(9-x-y) = 2x+4y-18$

$g_x = g_y = 0 \Rightarrow \left.\begin{array}{l} ① \ 4x+2y-18=0 \\ ② \ 2x+4y-18=0 \end{array}\right\} \xRightarrow{②\times 2} \left.\begin{array}{l} ① \ 4x+2y-18=0 \\ ② \ 4x+8y-36=0 \end{array}\right\} \xRightarrow{②-①} 6y = 18 \Rightarrow y=3 \Rightarrow x=3$

$\therefore x=3, y=3 \Rightarrow z = 9-x-y = 3$.

f clearly has an absolute min. which corresponds to an absolute min of g. The absolute min. of g is also a rel. min, so must occur at a C.P. We found one such C.P., so this CP must correspond to the absolute min. of f (subject to the constraint).

$\boxed{\therefore \ x=3, y=3, z=3 \text{ are the required numbers.}}$

[10]

Question 3: Find the volume of the solid that lies between the surface $z = y + \frac{x}{y^2}$ and the rectangle in the xy-plane $R = [0, 2] \times [1, 2]$.

$$V = \int_{y=1}^{2} \int_{x=0}^{2} \left(y + \frac{x}{y^2}\right) dx\, dy$$

$$= \int_{y=1}^{2} \left[xy + \frac{x^2}{2y^2}\right]_{x=0}^{2} dy$$

$$= \int_{y=1}^{2} \left(2y + \frac{4}{2y^2}\right) dy$$

$$= \left[y^2 - \frac{2}{y}\right]_1^2$$

$$= (4-1) - (1^2 - 2)$$

$$= 3 + 1$$

$$= \boxed{4}$$

[5]

Question 4: Evaluate the following double integral:

$$\int_0^1 \int_0^{y^2} y^3 e^{xy}\, dx\, dy$$

$$= \int_{y=0}^{1} \left[\cancel{y^3}^{\,2} \frac{e^{xy}}{\cancel{y}}\right]_{x=0}^{y^2} dy$$

$$= \int_0^1 \left(y^2 e^{y^3} - y^2\right) dy$$

$$= \left[\frac{e^{y^3}}{3} - \frac{y^3}{3}\right]_0^1$$

$$= \left(\frac{e}{3} - \frac{1}{3}\right) - \left(\frac{1}{3} - 0\right)$$

$$= \boxed{\frac{e-2}{3}}$$

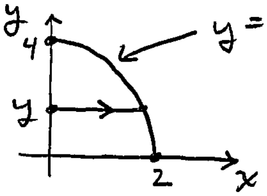
[5]

Question 5: Evaluate the following double integral:

$$I = \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx$$

(Hint: consider reversing the order of integration.)

D:



$y = 4-x^2 \Leftrightarrow x = \sqrt{4-y}$.

$$\therefore I = \int_{y=0}^{4} \int_{x=0}^{\sqrt{4-y}} \left(\frac{x\, e^{2y}}{4-y}\right) dx\, dy$$

$$= \int_{y=0}^{4} \left(\frac{e^{2y}}{4-y}\right) \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} dy$$

$$= \int_{y=0}^{4} \frac{e^{2y}}{\cancel{4-y}} \frac{\cancel{(\sqrt{4-y})^2}}{2}\, dy$$

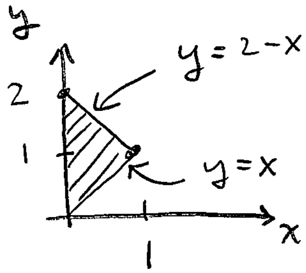
$$= \frac{1}{4}\left[e^{2y}\right]_0^4$$

$$= \boxed{\frac{1}{4}\left(e^8-1\right)}$$

[5]

Question 6: Find the volume of the region bounded between the paraboloid $z = x^2 + y^2$ and the triangle in the xy-plane enclosed by the lines $y = x$, $x = 0$ and $x + y = 2$.

D



$$V = \iint_D (x^2+y^2)\, dA$$

$$= \int_{x=0}^{1} \int_{y=x}^{2-x} (x^2+y^2)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$$

$$= \int_{x=0}^{1} x^2(2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\, dx$$

$$= \int_0^1 \left(-\frac{8}{3}x^3 + 4x^2 - 4x + \frac{8}{3}\right) dx$$

$$= \left[-\frac{2}{3}x^4 + \frac{4}{3}x^3 - 2x^2 + \frac{8}{3}x\right]_0^1$$

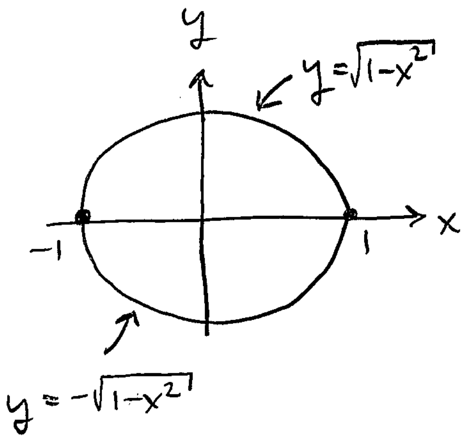
$$= -\frac{2}{3} + \frac{4}{3} - 2 + \frac{8}{3}$$

$$= \boxed{\frac{4}{3}}$$

[5]

Question 7: Compute

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx \;=\; I$$



$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \frac{2}{(1+r^2)^2}\, r\,dr\,d\theta \qquad \left[\begin{array}{l} u = 1+r^2 \\ du = 2r\,dr \end{array}\right.$$

$$= \int_{\theta=0}^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta$$

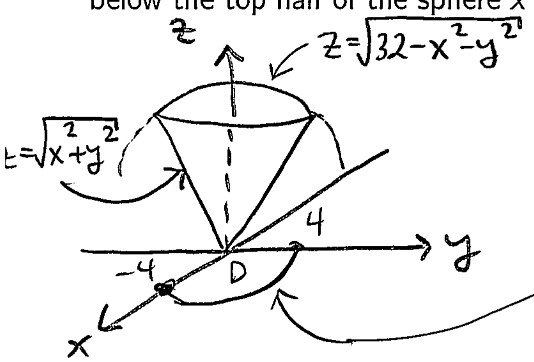
$$= \int_{\theta=0}^{2\pi} \left[-\frac{1}{2} - (-1)\right] d\theta$$

$$= (2\pi)\left(\frac{1}{2}\right)$$

$$= \boxed{\pi}$$

[5]

Question 8: Determine the volume in the first octant of the region lying above the cone $z = \sqrt{x^2+y^2}$ but below the top half of the sphere $x^2+y^2+z^2 = 32$.



Cone intersects sphere in 1st octant :

$$\left.\begin{array}{l} z = \sqrt{x^2+y^2} \\ x^2+y^2+z^2 = 32 \end{array}\right\} \Rightarrow x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 32 \Rightarrow x^2+y^2 = 16$$

$$\therefore\; V = \iint_D \left(\sqrt{32-x^2-y^2} - \sqrt{x^2+y^2}\right) dA$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{4} \left(\sqrt{32-r^2} - \sqrt{r^2}\right) r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{-(32-r^2)^{3/2}}{3} - \frac{r^3}{3}\right]_0^4 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(-\frac{128}{3} + \frac{128\sqrt{2}}{3}\right) d\theta$$

$$= \frac{128}{3}(\sqrt{2}-1)\cdot\frac{\pi}{2}$$

$$= \boxed{\frac{64\pi}{3}(\sqrt{2}-1)}$$

[5]