Question 1: Show that

$$\lim_{(x,y)\to(0,0)} -\frac{x}{\sqrt{x^2+y^2}}$$

does not exist. Clearly explain your reasoning.

- Let $(x,y) \to (0,0)$ along the positive x-axis (where $y=0$).
  Then $-\frac{x}{\sqrt{x^2+y^2}} = \frac{-x}{\sqrt{x^2}} = \frac{-x}{x} = -1 \to -1$ as $(x,y)\to(0,0)$.
- Let $(x,y) \to (0,0)$ along the positive y-axis (where $x=0$).
  Then $\frac{-x}{\sqrt{x^2+y^2}} = \frac{-0}{\sqrt{0^2+y^2}} = 0 \to 0$ as $(x,y)\to(0,0)$.

Since limiting values of $\frac{-x}{\sqrt{x^2+y^2}}$ differ along different approach paths, limit does not exist.

[5]

Question 2:

(a) Determine $\frac{\partial f}{\partial x}$ where $f(x) = \frac{1}{1+xy+(x/y)^2}$.

$$\boxed{f_x = \frac{-1}{\left[1+xy+\left(\frac{x}{y}\right)^2\right]^2}\cdot\left(y+\frac{2x}{y^2}\right)}$$

[2]

(b) Determine $g_y(0,1)$ where $g(x,y) = e^{xy}\ln(y)$.

$$g_y(x,y) = xe^{xy}\ln(y) + e^{xy}\frac{1}{y}$$

$$g_y(0,1) = 0e^0\ln(1) + e^0\cdot\frac{1}{1}$$

$$= \boxed{1}$$

[3]

Question 3: Let $f(x,y) = \sin^2(2x+ny)$ where n is a constant. Determine the value of n so that $f_{xy}(0,0) = 12$.

$$f_x = 2\sin(2x+ny)\cos(2x+ny)\cdot 2$$
$$= 4\sin(2x+ny)\cos(2x+ny)$$

$$f_{xy} = 4\cos(2x+ny)\cdot n\cdot\cos(2x+ny) + 4\sin(2x+ny)[-\sin(2x+ny)\cdot n]$$

$$f_{xy}(0,0) = 4n\underbrace{\cos^2(0)}_{1} - \underbrace{4n\sin^2(0)}_{0}$$
$$= 4n$$

$$f_{xy}(0,0) = 12 \Rightarrow \boxed{n=3}.$$

[5]

Question 4: Find $\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}$ if z is defined implicitly as a function of x and y by

$$xy + z^3 - 2yz = 0.$$

$$\frac{\partial}{\partial x}\left[xy+z^3-2yz\right] = \frac{\partial}{\partial x}[0]$$
$$y + 3z^2\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial x} = 0$$

At $(x,y,z) = (1,1,1)$:

$$1 + 3\cdot 1^2\cdot\frac{\partial z}{\partial x} - 2\cdot 1\cdot\frac{\partial z}{\partial x} = 0$$

$$1 + \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \boxed{-1}$$

[5]

Question 5: For this question consider the surface with equation

$$z = \frac{x-y}{xy+2} = f(x,y)$$

Use a linear approximation at $(x, y) = (1, -1)$ to approximate the value of z corresponding to $(x, y) = (1.1, -0.9)$.

$$L(x,y) = f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y+1)$$

$$= 2 + \left[\frac{(xy+2)(1) - (x-y)(y)}{(xy+2)^2}\right]\Bigg|_{(1,-1)}(x-1) + \left[\frac{(xy+2)(-1) - (x-y)(x)}{(xy+2)^2}\right]\Bigg|_{(1,-1)}(y+1)$$

$$= 2 + 3(x-1) - 3(y+1)$$

$$\therefore\ f(1.1,-0.9) \approx L(1.1,-0.9)$$

$$= L\left(1+\tfrac{1}{10}, -1+\tfrac{1}{10}\right)$$

$$= 2 + 3\left(\cancel{1}+\tfrac{1}{10}\cancel{-1}\right) - 3\left(\cancel{-1}+\tfrac{1}{10}\cancel{+1}\right)$$

$$= \boxed{2}$$

[5]

Question 6: Let $w = (x + y + z)^2$ where

$$x = r - s, \quad y = \cos(r + s), \quad z = \sin(r + s)$$

Compute $\partial w/\partial r$ at $(r, s) = (1, -1)$.

$$w_r = w_x x_r + w_y y_r + w_z z_r$$

$$= 2(x+y+z)(1) + 2(x+y+z)[-\sin(r+s)] + 2(x+y+z)\cos(r+s)$$

$$= 2(x+y+z)\left[1 - \sin(r+s) + \cos(r+s)\right].$$

At $(r,s) = (1,-1)$, $x = 1-(-1) = 2$, $y = \cos(1+(-1)) = 1$, $z = \sin(1+(-1)) = 0$.

$$\therefore\ w_r\Big|_{(r,s)=(1,-1)} = 2(2+1+0)\left[1 - \cancel{\sin(1+(-1))}^{\,0} + \cancel{\cos(1+(-1))}^{\,1}\right]$$

$$= \boxed{12}$$

[5]

Question 7: Find an equation of the tangent plane to the surface

$$\cos(\pi x) - x^2y + e^{xz} + yz = 4$$

at the point $(0, 1, 2)$.

Let $F(x,y,z) = \cos(\pi x) - x^2y + e^{xz} + yz$

$\nabla F(0,1,2) = \langle -\pi\sin(\pi x) - 2xy + ze^{xz}, -x^2+z, xe^{xz}+y \rangle \Big|_{(0,1,2)}$

$= \langle -\cancel{\pi\sin(0)}^{0} - 2\cdot0\cdot1 + 2e^0, -0^2+2, 0e^0+1 \rangle$

$= \langle 2, 2, 1 \rangle$.

Equation of tangent plane is

$\nabla F(0,1,2) \cdot \langle x-0, y-1, z-2 \rangle = 0$

$\langle 2,2,1 \rangle \cdot \langle x, y-1, z-2 \rangle = 0$

$$\boxed{\begin{array}{c} 2x + 2(y-1) + (z-2) = 0 \\ \text{or} \quad 2x+2y+z = 4 \end{array}}$$

[5]

Question 8: Suppose a smooth mountain has the shape of the paraboloid $z = 8 - 2x^2 - 3y^2$, and that a marble is released from rest at the point $(1, 1, 3)$ on the surface. Determine the unit tangent vector to the marble's initial direction of travel.

Let $f(x,y) = 8 - 2x^2 - 3y^3$. Marble will follow direction of steepest descent : $-\nabla f(1,1) = -\langle -4x, -6y \rangle \big|_{(1,1)} = \langle 4, 6 \rangle$.

The Δz in the tangent plane corresponding to $\Delta x = 4$ & $\Delta y = 6$ is

$\Delta z = f_x(1,1)\Delta x + f_y(1,1)\Delta y$

$= \nabla f(1,1) \cdot \langle \Delta x, \Delta y \rangle$

$= \langle -4, -6 \rangle \cdot \langle 4, 6 \rangle$

$= -52$

$\therefore$ A vector parallel to initial direction of travel is $\langle 4, 6, -52 \rangle$.

The unit vector is $\vec{u} = \dfrac{\langle 4,6,-52 \rangle}{|\langle 4,6,-52 \rangle|} = \dfrac{\langle 4,6,-52 \rangle}{2\sqrt{689}} = \boxed{\dfrac{\langle 2,3,-26 \rangle}{\sqrt{689}}}$

[5]

Question 9: Find all critical points of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$ and classify each as either a local maximum, local minimum or a saddle point. Carefully calculate all required derivatives and keep your work organized.

$$f_x(x,y) = -6x + 6y \quad ; \quad f_{xx}(x,y) = -6$$
$$f_y(x,y) = 6y - 6y^2 + 6x \quad ; \quad f_{yy}(x,y) = 6 - 12y$$
$$f_{xy} = 6$$

$f_x = 0 \Rightarrow -6x + 6y = 0 \Rightarrow x = y$

Set $x = y$ in $f_y = 0$: $6y - 6y^2 + 6y = 0$

$$12y - 6y^2 = 0$$
$$6y(2-y) = 0$$
$$y = 0 \quad ; \quad y = 2$$
$$\therefore \; x = 0 \quad ; \quad x = 2,$$

$\therefore$ Critical points are $(0,0)$ & $(2,2)$. (Note that f is a polynomial so $f_x(x,y)$ & $f_y(x,y)$ exist everywhere.)

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (-6)(6-12y) - 6^2.$$

CP	$D = -6(6-12y) - 36$	f_{xx}	Conclusion
$(0,0)$	-72	N/A	saddle point
$(2,2)$	72	-6	relative maximum

[10]