Question 1: Find the parametric equations of the line through the points $P(-2, 1, 3)$ and $Q(3, 2, -5)$.

Direction vector of line is $\vec{v} = \vec{PQ} = \langle 3-(-2), 2-1, -5-3 \rangle$

$= \langle 5, 1, -8 \rangle$

Vector equation of line is $\vec{r}(t) = \langle -2, 1, 3 \rangle + t \langle 5, 1, -8 \rangle$

$\therefore$ parametric equations are

$$\boxed{x = -2+5t, \quad y = 1+t, \quad z = 3-8t}$$

[5]

Question 2: Find the direction vector of a line that is perpendicular to both of the following lines:

$$L_1: \quad x = 3+2t, \quad y = 4-t, \quad z = 1+3t$$
$$L_2: \quad x = 1+4t, \quad y = 3-2t, \quad z = 4+5t$$

L_1 has direction $\vec{v} = \langle 2, -1, 3 \rangle$

L_2 has direction $\vec{u} = \langle 4, -2, 5 \rangle$

A direction vector $\perp$ to both $\vec{v}$ and $\vec{u}$ is

$$\vec{w} = \vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= \boxed{\langle 1, 2, 0 \rangle}$$

[5]

Question 3: Find an equation of the plane containing the points

$$P(1,2,3), \quad Q(0,-2,1) \text{ and } R(3,3,0)$$

normal to plane is $\vec{n} = \vec{PQ} \times \vec{PR}$

$$= \langle -1,-4,-2\rangle \times \langle 2,1,-3\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & -2 \\ 2 & 1 & -3 \end{vmatrix}$$

$$= \langle 14,-7,7\rangle = 7\langle 2,-1,1\rangle \Bigg\} \text{ use } \vec{n}_1 = \langle 2,-1,1\rangle \text{ for normal.}$$

using $P(1,2,3)$ and $\vec{n}_1 = \langle 2,-1,1\rangle$, equation of plane is

$$\langle x-1, y-2, z-3\rangle \cdot \langle 2,-1,1\rangle = 0$$

$$\Rightarrow 2(x-1) - (y-2) + (z-3) = 0$$

$$\Rightarrow \boxed{2x - y + z = 3}$$

[5]

Question 4: Find an equation of the plane that contains the line

$$\frac{x-2}{2} = \frac{y-3}{-2} = \frac{z}{7}$$

and is parallel to the plane $5x + 2y + z = 1$.

* Question is defective: Line intersects the plane, so a plane cannot contain the line and be parallel to $5x+2y+z=1$ at the same time!

Changing the line to, say,

$$\frac{x-2}{2} = \frac{y-3}{-2} = \frac{-z}{6}$$

would make the question consistent.

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Question 5: Find an equation of the plane which contains the line of intersection of the planes

$$x + y = 2 \text{ and } 2y - z = 3$$

and that is perpendicular to the plane $x - 2y + 3z = 1$.

Find two points on line of intersection which define a vector $\parallel$ to plane of interest : $x=0 \Rightarrow y=2 \Rightarrow z=1$: $P(0,2,1)$
$y=0 \Rightarrow x=2$, $z=-3$: $Q(2,0,-3)$

$\therefore \vec{v} = \vec{PQ} = \langle 2,-2,-4 \rangle$ is $\parallel$ to plane of interest.

Also, $x-2y+3z=1$ is $\perp$ to plane of interest,
so $\vec{u} = \langle 1,-2,3 \rangle$ is $\parallel$ to plane of interest.

$\therefore$ normal to plane is $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -2 & -4 \end{vmatrix} = \langle 14,10,2 \rangle = 2\langle 7,5,1 \rangle$

Using $\vec{n}_1 = \langle 7,5,1 \rangle$ and $P(0,2,1)$,
equation of plane is
$\langle x-0, y-2, z-1 \rangle \cdot \langle 7,5,1 \rangle = 0$
$\Rightarrow 7(x-0) + 5(y-2) + 1 \cdot (z-1) = 0 \quad \Rightarrow \boxed{7x + 5y + z = 11}$

[5]

Question 6: Find an equation of the plane having x-intercept a, y-intercept b and z-intercept c.

Points $P(a,0,0)$, $Q(0,b,0)$ and $R(0,0,c)$ are on plane.

$\therefore$ normal to plane is $\vec{PQ} \times \vec{PR} = \langle -a,b,0 \rangle \times \langle -a,0,c \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

$$= \langle bc, ac, ab \rangle$$
$$= \vec{n}$$

Using $P(a,0,0)$ then gives equation

$\langle x-a, y-0, z-0 \rangle \cdot \langle bc, ac, ab \rangle = 0$
$\Rightarrow bc(x-a) + ac(y-0) + ab(z-0) = 0$
$\Rightarrow \boxed{bcx + acy + abz = abc}$

[5]

Question 7: Let

$$\mathbf{r}_1(t) = \langle t^2, 7t-12, t^2\rangle \quad \text{and} \quad \mathbf{r}_2(t) = \langle 4t-3, t^2, 5t-6\rangle$$

represent the positions at time $t \geq 0$ of two particles travelling through space. When, if ever, do the particles collide?

Particles collide if $\vec{r}_1(t) = \vec{r}_2(t)$; comparing components:

x: $t^2 = 4t-3$
$t^2-4t+3=0$
$(t-1)(t-3)=0$
$t=1$, $\boxed{t=3}$

y: $7t-12=t^2$
$t^2-7t+12=0$
$(t-3)(t-4)=0$
$\boxed{t=3}$, $t=4$

z: $t^2=5t-6$
$t^2-5t+6=0$
$(t-3)(t-2)=0$
$\boxed{t=3}$, $t=2$

$\therefore$ corresponding components are equal and particles collide at $\boxed{t=3}$.

[5]

Question 8: Find parametric equations of the tangent line to $\mathbf{r}(t) = \langle 2-t^3, 2t-1, \ln(t)\rangle$ at the point $(1,1,0)$. (The tangent line is the line through the point in the direction of the tangent vector.)

$\vec{r}(t) = \langle 1,1,0\rangle$ at $t=1$.

$$\vec{r}\,'(1) = \frac{d}{dt}\langle 2-t^3, 2t-1, \ln(t)\rangle\Big|_{t=1}$$
$$= \langle -3t^2, 2, \tfrac{1}{t}\rangle\Big|_{t=1}$$
$$= \langle -3, 2, 1\rangle$$

$\therefore$ Tangent line is $\vec{\ell}(t) = \langle 1,1,0\rangle + t\langle -3,2,1\rangle$

$\therefore$
$$\boxed{\begin{aligned} x(t) &= 1-3t \\ y(t) &= 1+2t \\ z(t) &= t \end{aligned}}$$

[5]

Question 9: A particle starting at $(1, 0, 2)$ with velocity $3\mathbf{k}$ travels with acceleration $\mathbf{a}(t) = t\mathbf{i} + e^{-t}\mathbf{j}$ where $t \geq 0$ is time. Determine the position function $\mathbf{r}(t)$.

$\vec{a}(t) = \langle t, e^{-t}, 0 \rangle$, $\vec{v}(0) = \langle 0,0,3 \rangle$, $\vec{r}(0) = \langle 1,0,2 \rangle$.

$\therefore \vec{v}(t) = \int \langle t, e^{-t}, 0 \rangle \, dt = \langle \frac{t^2}{2}, -e^{-t}, 0 \rangle + \vec{C_1}$

$\vec{v}(0) = \langle 0,0,3 \rangle \Rightarrow \langle \frac{0^2}{2}, -e^{-0}, 0 \rangle + \vec{C_1} = \langle 0,0,3 \rangle$

$\therefore \vec{C_1} = \langle 0,0,3 \rangle - \langle 0,-1,0 \rangle$
$= \langle 0,1,3 \rangle$

$\therefore \vec{v}(t) = \langle \frac{t^2}{2}, -e^{-t}+1, 3 \rangle$

$\therefore \vec{r}(t) = \int \langle \frac{t^2}{2}, -e^{-t}+1, 3 \rangle \, dt = \langle \frac{t^3}{6}, e^{-t}+t, 3t \rangle + \vec{C_2}$

$\vec{r}(0) = \langle 1,0,2 \rangle \Rightarrow \langle \frac{0^3}{6}, e^{-0}+0, 3\cdot 0 \rangle + \vec{C_2} = \langle 1,0,2 \rangle$

$\therefore \vec{C_2} = \langle 1,0,2 \rangle - \langle 0,1,0 \rangle = \langle 1,-1,2 \rangle$

$\boxed{\therefore \vec{r}(t) = \langle \frac{t^3}{6}+1, e^{-t}+t-1, 3t+2 \rangle}$

[5]

Question 10: Determine all points of intersection of the space curve $\mathbf{r}(t) = \langle t, 1-t, 1/t^2 \rangle$ with the cylinder $x^2 + y^2 = 13$.

$x^2 + y^2 = 13$

$\therefore t^2 + (1-t)^2 = 13$

$\Rightarrow t^2 + 1 - 2t + t^2 = 13$

$2t^2 - 2t - 12 = 0$

$2(t^2 - t - 6) = 0$

$2(t-3)(t+2) = 0$

$\therefore t = 3, t = -2.$

$\therefore$ points of intersection are given by

$\vec{r}(3) = \langle 3, 1-3, \frac{1}{3^2} \rangle = \langle 3, -2, \frac{1}{9} \rangle$

and $\vec{r}(-2) = \langle -2, 1-(-2), \frac{1}{(-2)^2} \rangle = \langle -2, 3, \frac{1}{4} \rangle$

$\boxed{\therefore \text{points are } (3, -2, \tfrac{1}{9}) \;\&\; (-2, 3, \tfrac{1}{4})}$

[5]