Question 1:

(a) Give the equation of the plane through the point $(-3, 4, 7)$ that is parallel to the xy-plane.

$$\boxed{z = 7}$$

[2]

(b) The point (a, a, a) is located in the first octant (where $x \geq 0$, $y \geq 0$ and $z \geq 0$) at a distance 5 from the origin. Find the equation of the sphere of largest diameter that is located in the first octant and has (a, a, a) as the center.

Distance from (a,a,a) to $(0,0,0)$ is $\sqrt{a^2+a^2+a^2} = 5$

$\Rightarrow \sqrt{3a^2} = 5$

$\Rightarrow a = \frac{5}{\sqrt{3}}$ (note: $a \geq 0$ since (a,a,a) in 1^{st} octant)

$\therefore$ Centre of sphere is $\left(\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right)$ and maximum radius of sphere is $5/\sqrt{3}$.

$\therefore$ Equation is $\boxed{\left(x - \frac{5}{\sqrt{3}}\right)^2 + \left(y - \frac{5}{\sqrt{3}}\right)^2 + \left(z - \frac{5}{\sqrt{3}}\right)^2 = \frac{25}{3}}$

[3]

(c) Find an equation for the set of all points (x, y, z) that are equidistant from the points $(0, 0, 2)$ and $(1, 1, 1)$. Simplify your equation as much as possible.

$$\sqrt{(x-0)^2+(y-0)^2+(z-2)^2} = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$$

$$\Rightarrow \cancel{x^2} + \cancel{y^2} + \cancel{z^2} - 4z + 4 = \cancel{x^2} - 2x + 1 + \cancel{y^2} - 2y + 1 + \cancel{z^2} - 2z + 1$$

$$\boxed{2x + 2y - 2z = -1}$$

[5]

Question 2: For this question use the vectors

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

(a) Compute $|\mathbf{a} - 3\mathbf{c}|$.

$$\vec{a} - 3\vec{c} = \langle 2,-1,3\rangle - 3\langle 1,1,1\rangle = \langle -1,-4,0\rangle$$

$$\therefore |\vec{a} - 3\vec{c}| = \sqrt{(-1)^2 + (-4)^2 + 0^2} = \boxed{\sqrt{17}.}$$

[2]

(b) Compute $|a \times b|$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 3 & -2 & 1 \end{vmatrix} = 5\hat{i} + 7\hat{j} - \hat{k} .$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{5^2 + 7^2 + (-1)^2} = \sqrt{75} = \boxed{5\sqrt{3}} .$$

[2]

(c) Compute $\text{proj}_{\mathbf{a}}\mathbf{b}$.

$$\text{proj}_{\vec{a}}\vec{b} = \left(\vec{b} \cdot \frac{\vec{a}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$$

$$= \left(\langle 3,-2,1\rangle \cdot \frac{\langle 2,-1,3\rangle}{|\langle 2,-1,3\rangle|}\right)\frac{\langle 2,-1,3\rangle}{|\langle 2,-1,3\rangle|}$$

$$= \frac{11}{14}\langle 2,-1,3\rangle = \boxed{\left\langle \frac{11}{7}, \frac{-11}{14}, \frac{33}{14}\right\rangle}$$

[3]

(d) Find a vector of magnitude 5 that is parallel to **b** but pointing in the opposite direction.

Let $\vec{v}$ be the vector.

Then $\vec{v} = -5\dfrac{\vec{b}}{|\vec{b}|} = -5\dfrac{\langle 3,-2,1\rangle}{|\langle 3,-2,1\rangle|} = \dfrac{-5}{\sqrt{14}}\langle 3,-2,1\rangle$

$$= \left\langle \frac{-15}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{-5}{\sqrt{14}}\right\rangle$$

[3]

or

Question 3: Determine all values of x for which $\langle 3, 2, x\rangle$ and $\langle 2x, 4, x\rangle$ are orthogonal.

$$\langle 3,2,x\rangle \cdot \langle 2x,4,x\rangle = 0$$

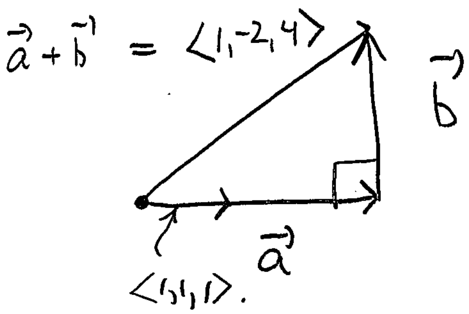
$$\Rightarrow 6x+8+x^2=0$$

$$\Rightarrow (x+2)(x+4)=0$$

$$\Rightarrow \boxed{x=-2,\ -4}$$

[5]

Question 4: Find orthogonal vectors **a** and **b** such that **a** is parallel to $\langle 1, 1, 1\rangle$ and $\mathbf{a}+\mathbf{b} = \langle 1, -2, 4\rangle$.



$$\therefore\ \vec{a} = \text{proj}_{\langle 1,1,1\rangle}\langle 1,-2,4\rangle$$

$$= \left(\langle 1,-2,4\rangle \cdot \frac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|}\right)\left(\frac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|}\right)$$

$$= \frac{3}{3}\langle 1,1,1\rangle$$

$$= \boxed{\langle 1,1,1\rangle}$$

$$\vec{b} = \langle 1,-2,4\rangle - \vec{a}$$

$$= \langle 1,-2,4\rangle - \langle 1,1,1\rangle$$

$$= \boxed{\langle 0,-3,3\rangle}$$

[5]

Question 5: Find all values of x such that the angle between $\langle 2, 1, -1\rangle$ and $\langle 1, x, 0\rangle$ is $\pi/4$.

$$\langle 2,1,-1\rangle \cdot \langle 1, x, 0\rangle = |\langle 2,1,-1\rangle|\,|\langle 1,x,0\rangle| \cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad 2+x = \sqrt{6}\sqrt{1+x^2}\cdot\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 4+4x+x^2 = 3(1+x^2)$$

$$\Rightarrow \quad 2x^2-4x-1=0$$

$$\Rightarrow \quad x = \frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-1)}}{2(2)}$$

$$= \frac{4\pm 2\sqrt{6}}{4}$$

$$= \boxed{\frac{2\pm\sqrt{6}}{2}}$$

[5]

Question 6: Find a unit normal vector to the plane containing the points $P(1, -3, -2)$, $Q(2, 0, -4)$ and $R(6, -2, -5)$.

Let $\vec{u}$ be the unit vector.

Then $\vec{u} = \dfrac{\vec{PQ}\times\vec{PR}}{|\vec{PQ}\times\vec{PR}|}$.

$$\vec{PQ}\times\vec{PR} = \langle 1, 3, -2\rangle \times \langle 5, 1, -3\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{vmatrix}$$

$$= \langle -7, -7, -14\rangle$$

$$= -7\langle 1,1,2\rangle.$$

$$\therefore \vec{u} = \frac{-7\langle 1,1,2\rangle}{|-7\langle 1,1,2\rangle|}$$

$$= \frac{-\langle 1,1,2\rangle}{\sqrt{6}}$$

$$= \boxed{\left\langle -\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right\rangle}$$

$\vec{u} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\rangle$ is correct also.

[5]

Question 7:

(a) Find the area of the triangle having vertices the points $P(1, -3, -2)$, $Q(2, 0, -4)$ and $R(6, -2, -5)$.

Area is $\frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}\,|-7\langle 1, 1, 2\rangle| = \boxed{\frac{7\sqrt{6}}{2}}$

[2]

(b) Is the triangle in part (a) a right triangle?

$|\vec{PQ}| = |\langle 1, 3, -2\rangle| = \sqrt{14}$

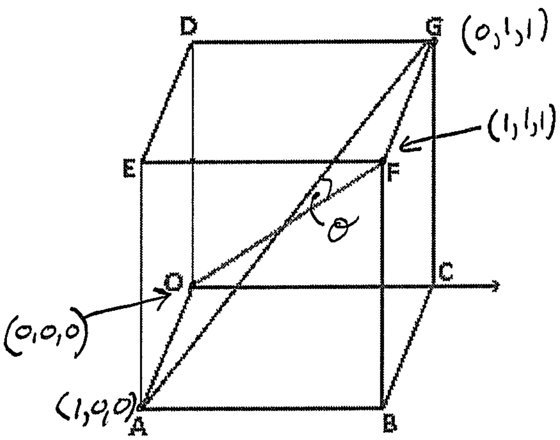
$|\vec{PR}| = |\langle 5, 1, -3\rangle| = \sqrt{35}$

$|\vec{QR}| = |\langle 4, -2, -1\rangle| = \sqrt{21}$

$|\vec{PQ}|^2 + |\vec{QR}|^2 = |\vec{PR}|^2$, so yes, triangle is a right triangle.

[3]

Question 8: Find the acute angle between the two diagonals OF and AG of a cube. If given a decimal answer, round your final answer to one decimal place.



$\vec{OF} = \langle 1, 1, 1\rangle$

$\vec{AG} = \langle 0, 1, 1\rangle - \langle 1, 0, 0\rangle = \langle -1, 1, 1\rangle.$

$(\vec{OF}) \cdot (\vec{AG}) = |\vec{OF}|\,|\vec{AG}| \cos(\theta)$

$\cancel{-1}\cancel{+1}+1 = \sqrt{3} \cdot \sqrt{3} \cos(\theta)$

$\therefore\ \theta = \cos^{-1}\left(\frac{1}{3}\right) \doteq$ $\boxed{\text{1.2 radians or 70.5 degrees.}}$

[5]