Question 1: Compute $\int \frac{10(2x+1)}{(x^2+4)(x+1)}\,dx = I$

$$\frac{10(2x+1)}{(x^2+4)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$= \frac{Ax^2+4A+Bx^2+Cx+Bx+C}{(x^2+4)(x+1)}$$

$$= \frac{(A+B)x^2+(B+C)x+4A+C}{(x^2+4)(x+1)}$$

$\therefore$ $A+B=0 \Rightarrow B=-A$

$B+C=20 \Rightarrow C = 20-B = 20+A$

$4A+C=10 \Rightarrow 4A+(20+A)=10 \Rightarrow 5A=-10$

$\therefore A=-2$

$B=2$

$C=18$

$$\therefore I = \int \frac{-2}{x+1} + \frac{2x+18}{x^2+4}\,dx$$

$$= \int \frac{-2}{x+1} + \underbrace{\frac{2x}{x^2+4}}_{\substack{u=x^2+4 \\ du=2x\,dx}} + \frac{18}{x^2+4}\,dx$$

$$= -2\ln|x+1| + \ln|x^2+4| + \frac{18}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$= \boxed{-2\ln|x+1| + \ln|x^2+4| + 9\arctan\left(\frac{x}{2}\right) + C}$$

[10]

Question 2: Compute $\displaystyle\int \frac{3}{x^2+4x+13}\,dx = I$

irreducible

$$x^2+4x+13 = (x+2)^2+9$$

$$\therefore I = \int \frac{3}{(x+2)^2+3^2}\,dx$$

$$= \frac{3}{3}\arctan\left(\frac{x+2}{3}\right)+C$$

$$= \boxed{\arctan\left(\frac{x+2}{3}\right)+C}$$

[5]

Question 3: Suppose the trapezoid rule is used to approximate $\int_0^2 e^{x-2}(x^2-4x+6)\,dx$. How many subintervals n are required to ensure that the error in the approximation is less than or equal to $1/24$. State your answer as a natural number.

$$f(x) = e^{x-2}(x^2-4x+6)$$

$$f'(x) = e^{x-2}(x^2-4x+6) + e^{x-2}(2x-4) = e^{x-2}(x^2-2x+2)$$

$$f''(x) = e^{x-2}(x^2-2x+2) + e^{x-2}(2x-2) = x^2e^{x-2}.$$

$$|f''(x)| = |x^2e^{x-2}| \leq 2^2e^{2-2} = 4 \text{ on } [0,2], \text{ so } K=4.$$

We require $\dfrac{K(b-a)^3}{12n^2} \leq \dfrac{1}{24}$

$$\Rightarrow \frac{4\cdot(2-0)^3}{12n^2} \leq \frac{1}{24}$$

$$\Rightarrow \frac{4\cdot 8\cdot \cancel{24}^2}{\cancel{12}} \leq n^2$$

$$\therefore 64 \leq n^2$$

$$\therefore 8 \leq n$$

$$\boxed{\therefore n=8}$$

[5]

Question 4: Determine whether the following improper integral is convergent or divergent. If convergent, evaluate the integral. Make proper use of any required limits:

$$I = \int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

$$I = \lim_{a \to 0^+} \int_a^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx \quad \left\} \begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\, dx \end{array} \right.$$

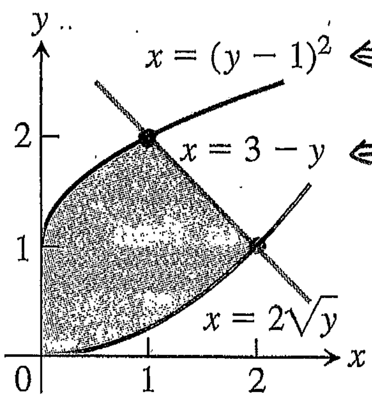
$$= \lim_{a \to 0^+} 2\left[e^{\sqrt{x}}\right]_a^1$$

$$= \lim_{a \to 0^+} 2\left[e^{\sqrt{1}} - e^{\sqrt{a}}\right]$$

$$= \boxed{2(e-1)}$$

[5]

Question 5: Determine the area of the shaded region:



$x = (y-1)^2 \iff y = 1 + x^{1/2}$

$x = 3 - y \iff y = 3 - x$

$x = 2\sqrt{y} \iff y = \frac{x^2}{4}$

$$A = \int_0^1 (1 + x^{1/2}) - \frac{x^2}{4}\, dx + \int_1^2 (3-x) - \frac{x^2}{4}\, dx$$

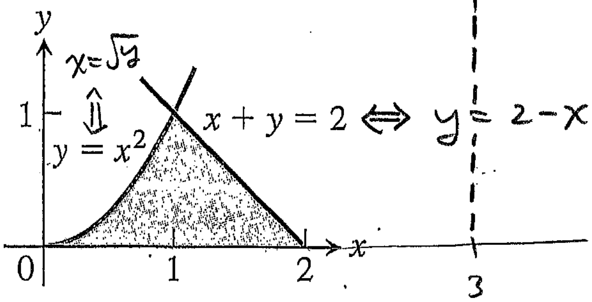
$$= \left[x + \frac{2}{3}x^{3/2} - \frac{x^3}{12}\right]_0^1 + \left[3x - \frac{x^2}{2} - \frac{x^3}{12}\right]_1^2$$

$$= 1 + \cancel{\frac{2}{3}} - \cancel{\frac{1}{12}} + 6 - 2 - \cancel{\frac{2}{3}} - 3 + \frac{1}{2} + \cancel{\frac{1}{12}}$$

$$= \boxed{\frac{5}{2}}$$

[5]

Question 6: For this question use the following region:



$x+y=2 \Leftrightarrow y = 2-x \Leftrightarrow x = 2-y$

(a) Determine the resulting volume if the region is rotated about the x-axis.

$$V = \int_0^1 \pi (x^2)^2 \, dx + \int_1^2 \pi (2-x)^2 \, dx$$

$$= \frac{\pi}{5}\left[x^5\right]_0^1 + \pi\left[4x - \frac{4}{2}x^2 + \frac{x^3}{3}\right]_1^2$$

$$= \frac{\pi}{5} + \pi\left[\cancel{8} - \cancel{8} + \frac{8}{3} - 4 + 2 - \frac{1}{3}\right]$$

$$= \pi\left[\frac{1}{5} + \frac{7}{3} - \frac{2}{1}\right] = \pi\left[\frac{3+35-30}{15}\right] = \boxed{\frac{8\pi}{15}}$$

[4]

(b) Determine the resulting volume if the region is rotated about the y-axis.

$$V = \int_{y=0}^{1} \pi (2-y)^2 - \pi (\sqrt{y})^2 \, dy$$

$$= \pi \int_0^1 (4 - 5y + y^2) \, dy$$

$$= \pi\left[4y - \frac{5}{2}y^2 + \frac{y^3}{3}\right]_0^1$$

$$= \pi\left[4 - \frac{5}{2} + \frac{1}{3}\right]$$

$$= \pi\left[\frac{24-15+2}{6}\right]$$

$$= \boxed{\frac{11\pi}{6}}$$

[4]

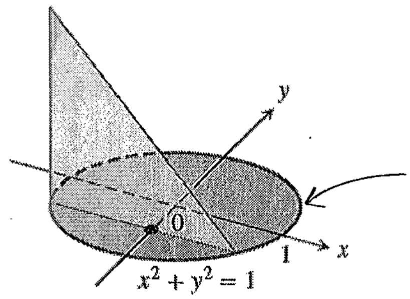
(c) Set up an integral representing the resulting volume if the region is rotated about the vertical line $x = 3$. DO NOT EVALUATE THIS LAST INTEGRAL, SIMPLY SET IT UP.

$$V = \int_{y=0}^{1} \pi (3-\sqrt{y})^2 - \pi (3-(2-y))^2 \, dy$$

$$= \pi \int_0^1 (3-\sqrt{y})^2 - (1+y)^2 \, dy$$

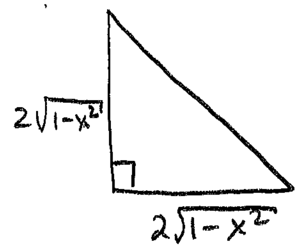
[2]

Question 7: A solid object has a circular base of radius 1. Cross-sections made perpendicular to the base are isosceles right triangles, as shown below. Determine the volume of the solid.



$y = \sqrt{1-x^2}$.

Triangle at slice position x is



$$\left.\begin{aligned} \text{area} &= \left(\tfrac{1}{\cancel{2}}\right)\left(\cancel{2}\sqrt{1-x^2}\right)\left(2\sqrt{1-x^2}\right) \\ &= 2(1-x^2). \end{aligned}\right.$$

$$\therefore V = \int_{-1}^{1} 2(1-x^2)\,dx$$

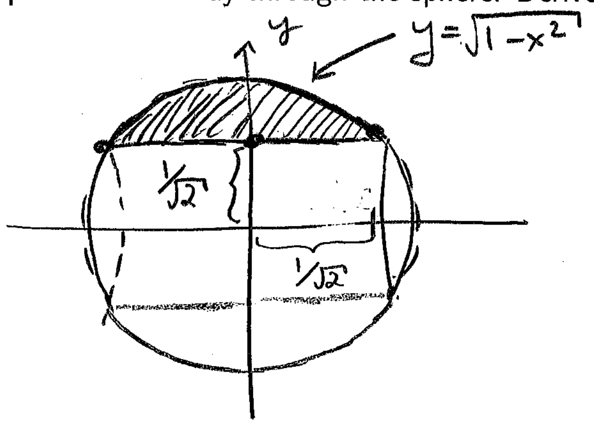
$$= 2\int_0^1 2(1-x^2)\,dx$$

$$= 4\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 4\left[1 - \frac{1}{3}\right] = \boxed{\frac{8}{3}}$$

[5]

Question 8: A sphere of radius 1 has a hole of radius $1/\sqrt{2}$ drilled through it. The hole is perfectly centred and passes all the way through the sphere. Derive the volume of the resulting object.



Object is generated by rotating indicated region about x-axis.

$$\therefore V = \int_{x=-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(\frac{1}{\sqrt{2}}\right)^2 dx$$

$$= 2\pi\int_0^{1/\sqrt{2}} 1 - x^2 - \frac{1}{2}\,dx$$

$$= 2\pi\left[\frac{1}{2}x - \frac{x^3}{3}\right]_0^{1/\sqrt{2}}$$

$$= 2\pi\left[\frac{1}{2\sqrt{2}} - \frac{1}{6\sqrt{2}}\right]$$

$$= \cancel{2}\pi\left(\frac{3-1}{3\cancel{6}\sqrt{2}}\right) = \boxed{\frac{\sqrt{2}\,\pi}{3}}$$

[5]