Question 1: The temperature $T(t)$ of an object over a 6 hour period is modeled by the function

$$T(t) = a + t(6 - kt)$$

where a and k are constants and $t = 0$ corresponds to the initial temperature measurement. Determine the values of a and k if the initial temperature of the object is 12 degrees and its average temperature over the 6 hour period is also 12 degrees.

$$T(0) = 12 \Rightarrow a + \cancel{0(6-k\cdot 0)} = 12$$

$$\Rightarrow a = 12$$

$$T_{ave} = 12 = \frac{1}{6-0}\int_0^6 T(t)\,dt$$

$$= \frac{1}{6}\int_0^6 12 + 6t - kt^2\,dt$$

$$= \frac{1}{6}\left[12t + \frac{6t^2}{2} - \frac{kt^3}{3}\right]_0^6$$

$$= \frac{1}{\cancel{6}}\left[(12)(\cancel{6}) + \frac{6^{\cancel{3}2}}{2} - \frac{6^{\cancel{3}2}k}{3}\right]$$

$$= 12 + 18 - 12k$$

$$\therefore \cancel{12} = \cancel{12} + 18 - 12k$$

$$k = \frac{18}{12} = \frac{3}{2}$$

$$\boxed{\therefore a = 12,\ k = \frac{3}{2}}$$

[5]

Question 2: Find $f(4)$ if $\displaystyle\int_0^x f(t)\,dt = x\cos(\pi x)$.

$$\frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}\left[x\cos(\pi x)\right]$$

$$f(x) = \cos(\pi x) - \pi x \sin(\pi x)$$

$$\therefore f(4) = \cos(4\pi) - \cancel{\pi\cdot 4\cdot\sin(4\pi)}$$

$$= \boxed{1}$$

[5]

Question 3: (Substitution Method)

(a) Determine $\int 3x\sqrt{7-3x^2}\,dx$. $= -\frac{1}{2}\int \sqrt{7-3x^2}\,(-6x)\,dx$

$u = 7-3x^2$

$du = -6x\,dx$

$= -\frac{1}{2}\int u^{1/2}\,du$

$= -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2} + C$

$= \boxed{-\frac{1}{3}(7-3x^2)^{3/2} + C}$

[2]

(b) Determine $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$. $= -\int \cos(u)\,du$

$u = \frac{1}{t} - 1$

$du = \frac{-1}{t^2}\,dt$

$= -\sin(u) + C$

$= \boxed{-\sin\left(\frac{1}{t}-1\right) + C}$

[2]

(c) Evaluate $\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$. $= 2\int_2^3 \frac{1}{u^2}\,du$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$x=1 \Rightarrow u=2$

$x=4 \Rightarrow u=3$

$= -2\left[\frac{1}{u}\right]_2^3$

$= -2\left[\frac{1}{3}-\frac{1}{2}\right]$

$= \boxed{\frac{1}{3}}$

[3]

(d) Determine $\int \sin(2\theta)e^{\sin^2(\theta)}\,d\theta$ $= \int e^{\sin^2\theta}\,2\sin\theta\cos\theta\,d\theta$

$\left.\begin{array}{l} u = \sin^2\theta \\ du = 2\sin\theta\cos\theta\,d\theta \end{array}\right\} \rightarrow = \int e^u\,du$

$= e^u + C$

$= \boxed{e^{\sin^2\theta} + C}$

[3]

Question 4: (Integration by Parts)

(a) Determine $\int (x^2+x+1)e^x\,dx$. $=I$

$u = x^2+x+1 \qquad dv = e^x\,dx$

$du = (2x+1)dx \qquad v = e^x$

$\therefore I = \int u\,dv$

$= uv - \int v\,du$

$= (x^2+x+1)e^x - \underbrace{\int e^x(2x+1)\,dx}$

$u = 2x+1 \qquad dv = e^x dx$

$du = 2dx \qquad v = e^x$

$= (x^2+x+1)e^x - \left[(2x+1)e^x - \int 2e^x\,dx\right]$

$= \boxed{(x^2+x+1)e^x - (2x+1)e^x + 2e^x + C}$

[5]

(b) Determine $\int \arctan(2x)\,dx$. $=I$

$u = \arctan(2x) \qquad dv = dx$

$du = \frac{2}{1+4x^2}dx \qquad v = x$

$I = \int u\,dv$

$= uv - \int v\,du$

$= x\arctan(2x) - \underbrace{\int \frac{2x}{1+4x^2}dx}$

$u = 1+4x^2$

$du = 8x\,dx$

$= x\arctan(2x) - \frac{1}{4}\int \frac{1}{u}du$

$= x\arctan(2x) - \frac{1}{4}\ln|u| + C$

$= \boxed{x\arctan(2x) - \frac{1}{4}\ln\left|1+4x^2\right| + C}$

[5]

Question 5: (Trigonometric Integrals)

(a) Evaluate $\int_0^{\pi/2} \sin^7(x)\cos^3(x)\,dx$. $= \int_0^{\frac{\pi}{2}} \sin^7(x)\,(1-\sin^2(x))\,\cos(x)\,dx$

$$\left.\begin{aligned} u &= \sin(x) \\ du &= \cos(x)\,dx \end{aligned}\right\} \begin{aligned} x=0 &\Rightarrow u=\sin(0)=0 \\ x=\tfrac{\pi}{2} &\Rightarrow u=\sin(\tfrac{\pi}{2})=1 \end{aligned}$$

$$\therefore I = \int_0^1 u^7(1-u^2)\,du$$

$$= \left[\frac{u^8}{8} - \frac{u^{10}}{10}\right]_0^1$$

$$= \frac{1}{8} - \frac{1}{10}$$

$$= \frac{2}{80} = \boxed{\frac{1}{40}}$$

[5]

(b) Determine $\int \sec^3(\theta)\,d\theta$.

(Hint: $\sec^3(\theta) = \sec(\theta)\sec^2(\theta)$, and integration by parts may be involved.)

$$I = \int \sec(\theta)\sec^2(\theta)\,d\theta \quad \left\}\begin{aligned} u &= \sec(\theta) & dv &= \sec^2(\theta)\,d\theta \\ du &= \sec(\theta)\tan(\theta)\,d\theta & v &= \tan(\theta) \end{aligned}\right.$$

$$= \int u\,dv$$

$$= uv - \int v\,du$$

$$= \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)\,d\theta$$

$$= \sec(\theta)\tan(\theta) - \int \sec(\theta)(\sec^2(\theta)-1)\,d\theta$$

$$= \sec(\theta)\tan(\theta) - \underbrace{\int \sec^3(\theta)\,d\theta}_{I} + \int \sec(\theta)\,d\theta$$

$$\therefore 2I = \sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|$$

$$\boxed{I = \frac{1}{2}\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right] + C}$$

[5]

Question 6: (Trigonometric Substitution) Determine

$$\mathcal{I} = \int \frac{\sqrt{x^2 - 36}}{x}\, dx$$

Let $x = 6\sec\theta$

$dx = 6\sec\theta\tan\theta\, d\theta$

$$\therefore \mathcal{I} = \int \frac{\sqrt{36(\sec^2\theta - 1)}}{\cancel{6\sec\theta}} \cdot \cancel{6\sec\theta}\tan\theta\, d\theta$$

$$= \int 6\tan\theta\tan\theta\, d\theta$$

$$= 6\int \sec^2\theta - 1\, d\theta$$

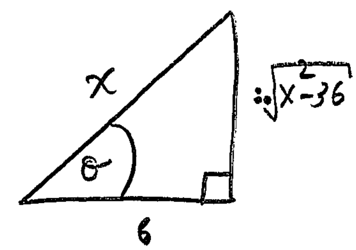
$$= 6\left[\tan\theta - \theta\right] + C$$

$$= 6\left[\frac{\sqrt{x^2-36}}{6} - \sec^{-1}\left(\frac{x}{6}\right)\right] + C$$

$$= \boxed{\sqrt{x^2-36} - 6\sec^{-1}\left(\frac{x}{6}\right) + C}$$

$x = 6\sec\theta$

$\therefore \sec\theta = \frac{x}{6}$;



[10]