Question 1: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{3^k (x-4)^k}{k^2}$$

$$u_k(x) = \frac{3^k (x-4)^k}{k^2}$$

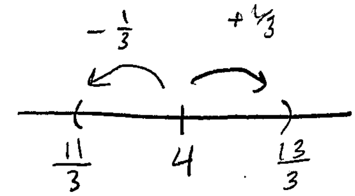
$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{3^{k+1}(x-4)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{3^k (x-4)^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{3^{k+1}}{3^k} \right| \cdot \underbrace{\left| \frac{k^2}{(k+1)^2} \right|}_{\to 1} \cdot \left| \frac{(x-4)^{k+1}}{(x-4)^k} \right| < 1$$

$$\Rightarrow 3\,|x-4| < 1$$

$$\Rightarrow |x-4| < \tfrac{1}{3}$$



$$\therefore\ \mathcal{I} = \left(\tfrac{11}{3}, \tfrac{13}{3}\right)$$

$$R = \tfrac{1}{3}$$

[5]

Question 2: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-3)^k}{k!5^k}$$

$$u_k(x) = \frac{(-1)^k (x-3)^k}{k!\,5^k}$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(x-3)^{k+1}}{(k+1)!\,5^{k+1}} \cdot \frac{k!\,5^k}{(-1)^k (x-3)^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \cancel{\left| \frac{(-1)^{k+1}}{(-1)^k} \right|} \cdot \left| \frac{k!}{(k+1)!} \right| \cdot \left| \frac{5^k}{5^{k+1}} \right| \cdot \left| \frac{(x-3)^{k+1}}{(x-3)^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{1}{k+1} \cdot \frac{1}{5} \cdot |x-3| < 1$$

$\Rightarrow 0 < 1$: true for all real x.

$$\therefore\ \mathcal{I} = (-\infty, \infty)$$

$$R = \infty$$

[5]

Question 3: A particle moves along a line with acceleration $a(t) = 10\sin(t) + 3\cos(t)$, initial position $s(0) = 0$ and position at time $t = 2\pi$ of $s(2\pi) = 12$. Determine $s(\pi/2)$, the position of the particle at time $t = \pi/2$. You may assume that position is measured in metres and time in seconds.

$$a(t) = 10\sin(t) + 3\cos(t)$$

$$\therefore v(t) = -10\cos(t) + 3\sin(t) + C_1$$

$$\therefore s(t) = -10\sin(t) - 3\cos(t) + C_1 t + C_2.$$

$$s(0) = 0 \Rightarrow -10\,\overset{0}{\cancel{\sin(0)}} - 3\,\overset{1}{\cancel{\cos(0)}} + C_1 \cdot 0 + C_2 = 0$$

$$\Rightarrow -3 + C_2 = 0$$

$$\Rightarrow C_2 = 3$$

$$\therefore s(t) = -10\sin(t) - 3\cos(t) + C_1 t + 3$$

$$s(2\pi) = 12 \Rightarrow -10\,\underset{0}{\cancel{\sin(2\pi)}} - 3\,\overset{1}{\cancel{\cos(2\pi)}} + C_1(2\pi) + 3 = 12$$

$$\therefore C_1 = \frac{12}{2\pi} = \frac{6}{\pi}$$

$$\therefore s\left(\frac{\pi}{2}\right) = -10\,\overset{1}{\cancel{\sin\left(\frac{\pi}{2}\right)}} - 3\,\overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}} + \left(\frac{6}{\pi}\right)\left(\frac{\pi}{2}\right) + 3$$

$$= \boxed{-4}$$

[5]

Question 4: Determine the most general antiderivative of each of the following:

(a) $f(x) = \dfrac{5x^4 - x^3 + 7x^2}{x^4} = 5 - \dfrac{1}{x} + 7x^{-2}$

$$\boxed{\therefore F(x) = 5x - \ln|x| - 7x^{-1} + C}$$

[2]

(b) $g(x) = \dfrac{\sec(x)\tan(x)}{4} - 3e^x + \pi^2 = \dfrac{1}{4}\sec(x)\tan(x) - 3e^x + \pi^2$

$$\boxed{\therefore G(x) = \frac{1}{4}\sec(x) - 3e^x + \pi^2 x + C}$$

[3]

Question 5: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_0^2 (2x - 4x^3)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$f(x) = 2x - 4x^3$

$[a,b] = [0,2]$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_i = 0 + i\Delta x = \frac{2i}{n}$

$f(x_i) = 2\left(\frac{2i}{n}\right) - 4\left(\frac{2i}{n}\right)^3 = \frac{4i}{n} - \frac{32i^3}{n^3}$

$$\int_0^2 (2x-4x^3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n} - \frac{32i^3}{n^3}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^2}\left(\sum_{i=1}^{n} i\right) - \frac{64}{n^4}\left(\sum_{i=1}^{n} i^3\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^2} \cdot \frac{n(n+1)}{2} - \frac{64}{n^4} \frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{2} \cdot \frac{n}{n} \cdot \frac{n+1}{n} - \frac{64}{4} \cdot \frac{n}{n} \cdot \frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{n+1}{n}\right]$$

$$= 4 - 16$$

$$= \boxed{-12}$$

[Check: $\int_0^2 (2x-4x^3)\,dx = \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{\cancel{4}x^4}{\cancel{4}}\right]_0^2 = (2^2 - 2^4) - (0^2 - 0^4)$
$= -12$ ✓]

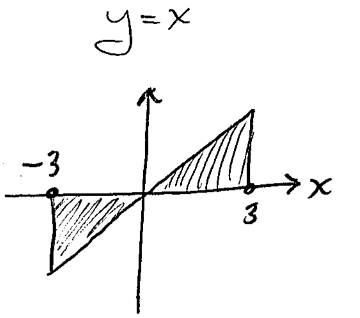
[10]

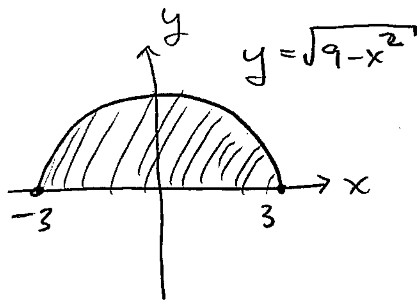
Question 6: Use an area interpretation to evaluate $\int_{-3}^{3} (x - \sqrt{9 - x^2})\, dx$.

(Hint: split up the integrand and sketch some graphs.)

Over $[-3, 3]$:

$y = x$





$\therefore \int_{-3}^{3} x\, dx = 0$

$\therefore \int_{-3}^{3} \sqrt{9-x^2}\, dx = \left(\frac{1}{2}\right) \pi \cdot 3^2 = \frac{9}{2}\pi$

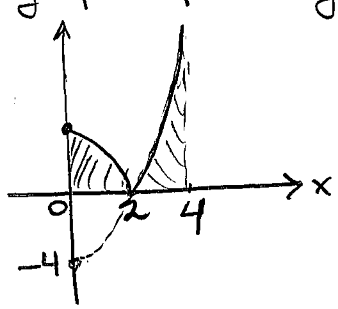
$$\therefore \int_{-3}^{3} (x - \sqrt{9-x^2})\, dx = \cancel{\int_{-3}^{3} x\, dx}^{\,0} - \int_{-3}^{3} \sqrt{9-x^2}\, dx$$

$$= \boxed{-\frac{9}{2}\pi}$$

[5]

Question 7: Determine $\int_0^4 |x^2 - 4|\, dx$

$y = |x^2 - 4|$ has graph



$$\therefore \int_0^4 |x^2-4|\, dx = \int_0^2 -(x^2-4)\, dx + \int_2^4 (x^2-4)\, dx$$

$$= -\left[\frac{x^3}{3} - 4x\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^4$$

$$= -\left[\frac{8}{3} - 8\right] + \left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right]$$

$$= -\frac{8}{3} - \frac{8}{3} + \frac{64}{3} \cancel{+8+8-16}$$

$$= \frac{48}{3}$$

$$= \boxed{16}$$

[5]

Question 8: Determine the following integrals:

(a) $\int_1^2 \frac{(1-x)(1+x)}{x}\,dx = \int_1^2 \frac{1-x^2}{x}\,dx$

$= \int_1^2 \frac{1}{x} - x\,dx$

$= \left[\ln|x| - \frac{x^2}{2}\right]_1^2$

$= \left(\ln|2| - \frac{2^2}{2}\right) - \left(\cancel{\ln|1|}^{0} - \frac{1^2}{2}\right)$

$= \boxed{\ln(2) - \frac{3}{2}}$

[3]

(b) $\int 4x^3 - 3^x\,dx = \frac{\cancel{4}x^4}{\cancel{4}} - \frac{3^x}{\ln(3)} + C$

$= \boxed{x^4 - \frac{3^x}{\ln(3)} + C}$

[2]

(c) $\int_{\pi/4}^{\pi/2} \csc^2(x)\,dx = \left[-\cot(x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -\cancel{\cot\left(\frac{\pi}{2}\right)}^{0} - \left[-\cancel{\cot\left(\frac{\pi}{4}\right)}^{1}\right]$

$= \boxed{1}$

[2]

(d) $\int_0^\pi 2e^t - \frac{\cos(t)}{\pi}\,dt$

$= \left[2e^t - \frac{\sin(t)}{\pi}\right]_0^\pi$

$= \left(2e^\pi - \cancel{\frac{\sin(\pi)}{\pi}}^{0}\right) - \left(2\cancel{e^0}^{1} - \cancel{\frac{\sin(0)}{\pi}}^{0}\right)$

$= \boxed{2(e^\pi - 1)}$

[3]