Question 1:

(a) Determine the linear approximation $T_1(x)$ for $f(x) = e^{-x} + e^{-2x}$ at $a = 0$ and use it to approximate $f(0.2)$. Simplify your final answer.

$T_1(x) = f(0) + f'(0)x$.

$f(0) = e^{-0} + e^{-2\cdot 0} = 2$

$f'(x) = -e^{-x} - 2e^{-2x}$

$f'(0) = -e^{-0} - 2e^{-2\cdot 0} = -1 - 2 = -3$

$\therefore T_1(x) = f(0) + f'(0)x$

$= 2 - 3x$

$\therefore f(0.2) = f(\frac{1}{5}) \approx T_1(\frac{1}{5}) = 2 - 3(\frac{1}{5}) = \boxed{\frac{7}{5}}$

[5]

(b) Give an error bound on your approximation in part (a). Again, simplify your final answer.

$|R_1(x)| = \left| \frac{f''(z)}{2} x^2 \right|$ where $a = 0$, $x = \frac{1}{5}$, $0 < z < \frac{1}{5}$

$f'(z) = -e^{-z} - 2e^{-2z}$

$f''(z) = e^{-z} + 4e^{-2z}$

$\therefore |R_1(\frac{1}{5})| = \left| \frac{(e^{-z} + 4e^{-2z})}{2} \cdot \left(\frac{1}{5}\right)^2 \right|$

$\leq \left| \frac{(e^{-0} + 4e^{-2\cdot 0})}{2} \frac{1}{5^2} \right|$

$= \left(\frac{5}{2}\right)\left(\frac{1}{5^2}\right) = \boxed{\frac{1}{10}}$

[5]

Question 2:

(a) Use a Taylor polynomial of degree 2 to approximate 1/1.1. Simplify your final answer.

Use $f(x) = \frac{1}{x}$, $a = 1$

$f(a) = f(1) = \frac{1}{1} = 1$

$f'(x) = \frac{-1}{x^2}$; $f'(a) = f'(1) = \frac{-1}{1^2} = -1$

$f''(x) = \frac{2}{x^3}$; $f''(a) = f''(1) = \frac{2}{1^3} = 2$

$\therefore T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$= 1 + (-1)(x-1) + \frac{2}{2}(x-1)^2$

$= 1 - (x-1) + (x-1)^2$

$\therefore f(1.1)$

$\approx T_2(1.1)$

$= 1 - (1.1-1) + (1.1-1)^2$

$= 1 - \frac{1}{10} + \frac{1}{10^2}$

$= \boxed{\frac{91}{100}}$

[5]

(b) Give an error bound on your approximation in part (a). Again, simplify your final answer.

$|R_2(x)| = \left| \frac{f'''(z)}{3!}(x-a)^3 \right|$ where $a=1$, $x = 1.1$, $1 < z < 1.1$

$f'''(z) = \frac{-6}{z^4}$

$\therefore |R_2(1.1)| = \left| \frac{1}{3!} \cdot \frac{-6}{z^4} \cdot (1.1-1)^3 \right|$

$\leq \left| \frac{1}{6} \cdot \frac{-6}{1^4} \cdot \frac{1}{10^3} \right|$

$= \boxed{\frac{1}{1000}}$

[5]

Question 3:
Find the Taylor series about $a = -2$ for $f(x) = 1 + x + 2x^2 + 3x^3$. You should be able to write all terms of the series.

$f(x) = 1 + x + 2x^2 + 3x^3$; $f(-2) = 1 - 2 + 2(-2)^2 + 3(-2)^3 = -17$

$f'(x) = 1 + 4x + 9x^2$; $f'(-2) = 1 + 4(-2) + 9(-2)^2 = 29$

$f''(x) = 4 + 18x$; $f''(-2) = 4 + 18(-2) = -32$

$f'''(x) = 18$; $f'''(-2) = 18$

$f^{(k)}(x) = 0$ for all $k \geq 4$.

$$\therefore T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \boxed{-17 + 29(x+2) - 16(x+2)^2 + 3(x+2)^3}$$

[5]

Question 4: Find the first four nonzero terms of the Taylor series about $a = 1$ for $g(x) = \dfrac{3}{2+4x}$ and state the open interval of convergence.

$$\frac{3}{2+4x} = \frac{3}{2+4(x-1)+4}$$

$$= \frac{3}{6+4(x-1)}$$

$$= \frac{1}{2} \frac{1}{1 - \left[-\frac{2}{3}(x-1)\right]}$$

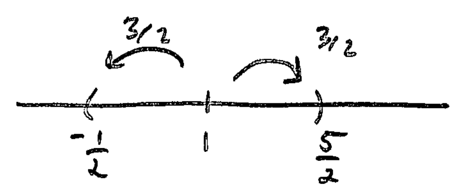
$$= \frac{1}{2}\left[1 + \left[-\frac{2}{3}(x-1)\right] + \left[-\frac{2}{3}(x-1)\right]^2 + \cdots\right]$$

$$= \boxed{\frac{1}{2} - \frac{1}{3}(x-1) + \frac{2}{3^2}(x-1)^2 - \frac{2^2}{3^3}(x-1)^3 + \cdots}$$

Converges for

$$\left|-\frac{2}{3}(x-1)\right| < 1$$

$$\Rightarrow |x-1| < \frac{3}{2}$$



$$\boxed{\therefore I = \left(-\frac{1}{2}, \frac{5}{2}\right)}$$

[5]

Question 5: Find the Maclaurin polynomial of degree 7 for $f(x) = \dfrac{\ln(1+x^2)}{x}$.

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots$$

$$\therefore \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$$

$$\therefore \frac{\ln(1+x^2)}{x} = x - \frac{x^3}{2} + \frac{x^5}{3} - \frac{x^7}{4} + \cdots$$

$$\boxed{\therefore T_7(x) = x - \frac{x^3}{2} + \frac{x^5}{3} - \frac{x^7}{4}}$$

[3]

Question 6: Find the sum of the infinite series $1 - \dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} - \cdots$.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\therefore 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \cdots = \cos(\pi) = \boxed{-1}$$

[3]

Question 7: Find the first three non-zero terms of the Maclaurin series for $f(x) = e^{-x}\ln(1+x)$.

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots$$

$$\therefore e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

	1	$-x$	$+\frac{x^2}{2}$	$-\frac{x^3}{3!}$	$+\cdots$
x	x	$-x^2$	$+\frac{x^3}{2}$	$-\frac{x^4}{3!}$	$+\cdots$
$-\frac{x^2}{2}$	$-\frac{x^2}{2}$	$+\frac{x^3}{2}$	$-\frac{x^4}{4}$	$+\frac{x^5}{2\cdot 3!}$	
$+\frac{x^3}{3}$	$\frac{x^3}{3}$	$-\frac{x^4}{3}$	$+\frac{x^5}{6}$	$-\frac{x^6}{3\cdot 3!}$	
$-\frac{x^4}{4}$	$-\frac{x^4}{4}$	$+\frac{x^5}{5}$	$-\frac{x^6}{8}$	$+\frac{x^7}{4\cdot 3!}$	
$\vdots$					

$$\therefore T(x) = x - x^2 - \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^3}{2} + \frac{x^3}{3} + \cdots$$

$$= \boxed{x - \frac{3}{2}x^2 + \frac{4}{3}x^3 + \cdots}$$

[4]

Question 8: Use series to find the limit: $\lim_{x\to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$

$$= \lim_{x\to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \cdots\right) - \cancel{x} + \cancel{\frac{x^3}{6}}}{x^5}$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}{x^5}$$

$$= \lim_{x\to 0} \frac{\cancel{x^5}\left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots\right)}{\cancel{x^5}}$$

$$= \boxed{\frac{1}{5!}}$$

[5]

Question 9: It can be shown that

$$\cos[\sin(x)] = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{37x^6}{720} + \cdots$$

for all real numbers x. Use this fact to find the first 3 non-zero terms of the Maclaurin series for $g(x) = \sin[\sin(x^2)]\cos(x^2)$.

$$\frac{d}{dx}\cos\left[\sin(x)\right] = \frac{d}{dx}\left[1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{37x^6}{720} + \cdots\right]$$

$$-\sin\left[\sin(x)\right]\cos(x) = -x + \frac{5}{6}x^3 - \frac{37}{120}x^5 + \cdots$$

$$\therefore \sin\left[\sin(x)\right]\cos(x) = x - \frac{5}{6}x^3 + \frac{37}{120}x^5 + \cdots$$

$$\therefore \sin\left[\sin(x^2)\right]\cos(x^2) = \boxed{x^2 - \frac{5}{6}x^6 + \frac{37}{120}x^{10} - \cdots}$$

[5]