Question 1: Find the following derivatives (Take care to use proper notation; it is not necessary to simplify your answers):

(a) $f(x) = x^2 e^{-2/x}$

$$f'(x) = 2x e^{-\frac{2}{x}} + x^2 e^{-\frac{2}{x}} \left(\frac{2}{x^2}\right)$$

$$= 2xe^{-\frac{2}{x}} + 2e^{-\frac{2}{x}}$$

$$= \boxed{2e^{-\frac{2}{x}}(x+1)}$$

[2]

(b) $y = \ln(\sec^2 x)$ $= 2\ln(\sec x)$

$$y' = \frac{2}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x$$

$$= \boxed{2\tan x}$$

[2]

(c) $g(t) = t\arctan(2t) - 2^t$

$$\boxed{g'(t) = \arctan(2t) + \frac{2t}{1+4t^2} - 2^t \ln(2)}$$

[3]

(d) $y = \log_5(1 + e^{\ln(x^2)})$ $= \log_5(1+x^2)$

$$y' = \frac{1}{(1+x^2)\ln 5} \cdot 2x \quad = \boxed{\frac{1}{\ln(5)} \frac{2x}{1+x^2}}$$

[3]

Question 2: The equation $x^y = e$ defines y implicitly as a function of x. Find the equation of the tangent line to the corresponding curve at the point $(e, 1)$. (Hint: y can be expressed explicitly as a function of x.)

$$x^y = e$$

$$\ln(x^y) = \ln(e)$$

$$y \ln(x) = 1$$

$$y = \frac{1}{\ln(x)}$$

$$y' = \frac{-1}{x[\ln(x)]^2}$$

$$y'\Big|_{x=e} = \frac{-1}{e[\cancel{\ln(e)}]^2}$$

$$= \frac{-1}{e}$$

$$\boxed{\therefore\ y - 1 = \frac{-1}{e}(x - e)}$$

[5]

Question 3: Suppose $f(x) = x^3 + 3\sin(x) + 2\cos(x)$. Find $(f^{-1})'(2)$, the derivative of the inverse function at $x = 2$.

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

$f^{-1}(2)$ is the solution to $x^3 + 3\sin(x) + 2\cos(x) = 2$

$$\Rightarrow \quad x = 0$$

So $f'(f^{-1}(2)) = f'(0) = \frac{d}{dx}\left[x^3 + 3\sin(x) + 2\cos(x)\right]\Big|_{x=0}$

$$= \left[3x^2 + 3\cos(x) - 2\sin(x)\right]\Big|_{x=0}$$

$$= 3$$

$$\therefore\ (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \boxed{\frac{1}{3}}$$

[5]

Question 4: Evaluate the following limits, if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning.

(a) $\displaystyle\lim_{x\to 0}\frac{8x^2}{\cos(x)-1}$ $\sim$ "$\frac{0}{0}$"

$\overset{H}{=} \displaystyle\lim_{x\to 0}\frac{16x}{-\sin(x)}$ $\sim$ "$\frac{0}{0}$"

$\overset{H}{=} \displaystyle\lim_{x\to 0}\frac{16}{-\cos(x)} = \boxed{-16}$

[3]

(b) $\displaystyle\lim_{x\to 0^+}\frac{\ln(e^x-1)}{\ln(x)}$ $\sim$ "$\frac{-\infty}{-\infty}$"

$\overset{H}{=} \displaystyle\lim_{x\to 0^+}\frac{\left(\frac{e^x}{e^x-1}\right)}{\frac{1}{x}}$

$= \displaystyle\lim_{x\to 0^+}\frac{xe^x}{e^x-1}$ $\sim$ "$\frac{0}{0}$"

$\overset{H}{=} \displaystyle\lim_{x\to 0^+}\frac{e^x+xe^x}{e^x} = \boxed{1}$

[3]

(c) $\displaystyle\lim_{x\to 1^+} x^{1/(1-x)}$ $\sim$ "$1^{-\infty}$"

$= \displaystyle\lim_{x\to 1^+} e^{\left(\frac{1}{1-x}\right)\ln(x)}$

$= \displaystyle\lim_{x\to 1^+} e^{\frac{\ln(x)}{1-x}}$

For $\frac{\ln(x)}{1-x}$:

$\displaystyle\lim_{x\to 1^+}\frac{\ln(x)}{1-x} =$ "$\frac{0}{0}$"

$\overset{H}{=} \displaystyle\lim_{x\to 1^+}\frac{(1/x)}{-1} = -1$

$\therefore \displaystyle\lim_{x\to 1^+} x^{\frac{1}{1-x}}$

$= \boxed{e^{-1}}$

[4]

Question 5: Determine the absolute minimum and maximum values of $f(x) = x^4 - 2x^2 - 3$ on the interval $[-2, 2]$

$$f'(x) = 4x^3 - 4x$$
$$= 4x(x^2-1)$$

• $f'(x) = 0$? $x = 0, 1, -1$

• $f'(x)$ not exist? no such x.

x	$f(x) = x^4 - 2x^2 - 3$
-2	5
-1	-4
0	-3
1	-4
2	5

$\therefore$ f has an $\boxed{\text{abs. max. of } 5}$ at $x = -2$ & $x = 2$

f has an $\boxed{\text{abs. min. of } -4}$ at $x = -1$ & $x = 1$

[10]

Question 6: For this question let $f(x) = (x^2 - 4)^{2/3}$ $\}$ Domain is $(-\infty, \infty)$.

(a) Determine the intervals on which f is increasing or decreasing.

$$f'(x) = \frac{2}{3}(x^2-4)^{-1/3}(2x) = \frac{4x}{3(x^2-4)^{1/3}}.$$

- $\underline{f'(x)=0}$? $x=0$.
- $\underline{f'(x) \text{ not exist}}$? $x = 2, -2$.

Crit. num:		-2		0		2	
test pts:	3		-1		1		3
$f'(x) = \frac{4x}{3(x^2-4)^{1/3}}$:	$-$	NA	$+$	0	$-$	NA	$+$
$f(x) = (x^2-4)^{2/3}$:	$\searrow$	0	$\nearrow$	$4^{2/3}$	$\searrow$	0	$\nearrow$

$\therefore$ f is decreasing on $(-\infty, -2) \cup (0, 2)$.

f is increasing on $(-2, 0) \cup (2, \infty)$.

[8]

(b) Determine the local (or relative) maximum and minimum values of f.

f has a rel. min. of 0 at $x = -2$,

a rel. max. of $4^{2/3}$ at $x = 0$,

a rel. min. of 0 at $x = 2$.

[2]