

Question 1 [12 points]: Evaluate the following limits (you may use the fact that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$):

$$(a) \quad \lim_{x \rightarrow \infty} \frac{2e^x - 7e^{-x} + 3}{5e^x + 4e^{-x} - 1} = \lim_{x \rightarrow \infty} \frac{2 - 7e^{-2x} + 3e^{-x}}{5 + 4e^{-2x} - e^{-x}}$$

$$= \frac{2}{5}$$

$$(b) \quad \lim_{x \rightarrow 5} \frac{x^2 - 7x + 10}{x^2 - 8x + 15} = \lim_{x \rightarrow 5} \frac{(x-5)(x-2)}{(x-5)(x-3)} = \frac{3}{2}$$

$$(c) \quad \lim_{x \rightarrow 0} \frac{\tan(3x) \cos(2x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \cdot \frac{3}{\cos(3x)} \cdot \cos(2x)$$

$$= 3$$

Question 2 [8 points]:

- (a) Let $f(x) = \sqrt{x+1}$. Use the definition of the derivative to find $f'(x)$. (No credit will be given if $f'(x)$ is found using differentiation rules.)

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} (\sqrt{x+h+1} - \sqrt{x+1}) \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{x+h+1 - x-1}{\sqrt{x+h+1} + \sqrt{x+1}} \\
 &= \frac{1}{2\sqrt{x+1}}
 \end{aligned}$$

- (b) Find the linear approximation to $f(x) = \sqrt{x+1}$ at $a = 3$.

$$f(x) = \sqrt{x+1} ; f(3) = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} ; f'(3) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\therefore L(x) = f(a) + f'(a)(x-a)$$

$$= 2 + \frac{1}{4}(x-3)$$

$$= \frac{1}{4}x + \frac{5}{4}$$

Question 3 [15 points]: Differentiate the following functions (you do not need to simplify your answers):

(a) $y = 5 \cos(x^7 - 8x)$

$$y' = -5 \sin(x^7 - 8x) \cdot (7x^6 - 8)$$

(b) $y = e^x \cot x$

$$y' = e^x \cot x - e^x \csc^2 x$$

(c) $y = \frac{2\sqrt{x}}{e^x - \pi x}$ (Use the Quotient Rule.)

$$\begin{aligned} y' &= \frac{(e^x - \pi x) \cdot 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} - 2\sqrt{x} (e^x - \pi)}{(e^x - \pi x)^2} \\ &= \frac{(e^x - \pi x) x^{-\frac{1}{2}} - 2x^{\frac{1}{2}} (e^x - \pi)}{(e^x - \pi x)^2} \end{aligned}$$

(d) $y = x^3 4^{\tan x}$

$$y' = 3x^2 4^{\tan x} + x^3 4^{\tan x} \cdot \ln 4 \cdot \sec^2 x$$

(e) $y = \sqrt[5]{\log_7(x^2 - \cos x)} = \left[\log_7(x^2 - \cos x) \right]^{\frac{1}{5}}$

$$y' = \frac{1}{5} \left[\log_7(x^2 - \cos x) \right]^{-\frac{4}{5}} \cdot \frac{1}{(x^2 - \cos x) \ln 7} \cdot (2x + \sin x)$$

Question 4 [10 points]:

(a) Find the general antiderivative of the following functions:

(i) $f(x) = \sec^2 x + 3 \sin x - \frac{e^x}{2}$

$$F(x) = \tan x - 3 \cos x - \frac{1}{2} e^x + C$$

(ii) $y = \frac{2x^4 - \sqrt[3]{x^2} + 7}{x} = 2x^3 - x^{-\frac{1}{3}} + 7x^{-1}$

$$\begin{aligned} \therefore F(x) &= \frac{2x^4}{4} - \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + 7 \ln|x| + C \\ &= \frac{x^4}{2} - \frac{3}{2} x^{\frac{2}{3}} + 7 \ln|x| + C \end{aligned}$$

(b) Find the function $f(t)$ with $f'(t) = 4t^3 + 2t$ and $f(1) = 9$.

$$f(t) = \frac{4t^4}{4} + \frac{2t^2}{2} + C = t^4 + t^2 + C$$

$$f(1) = 9 \Rightarrow 1 + 1 + C = 9$$

$$\therefore C = 7$$

$$\therefore f(t) = t^4 + t^2 + 7.$$

Question 5 [10 points]:

(a) Find $\frac{dy}{dx}$ by implicit differentiation:

$$xy + e^{3x} = \sin(x + y)$$

$$y + xy' + 3e^{3x} = \cos(x + y)(1 + y')$$

$$y + xy' + 3e^{3x} = \cos(x + y) + y' \cos(x + y)$$

$$y'(x - \cos(x + y)) = \cos(x + y) - y - 3e^{3x}$$

$$\therefore y' = \frac{\cos(x + y) - y - 3e^{3x}}{x - \cos(x + y)}$$

(b) Use logarithmic differentiation to find $\frac{dy}{dx}$:

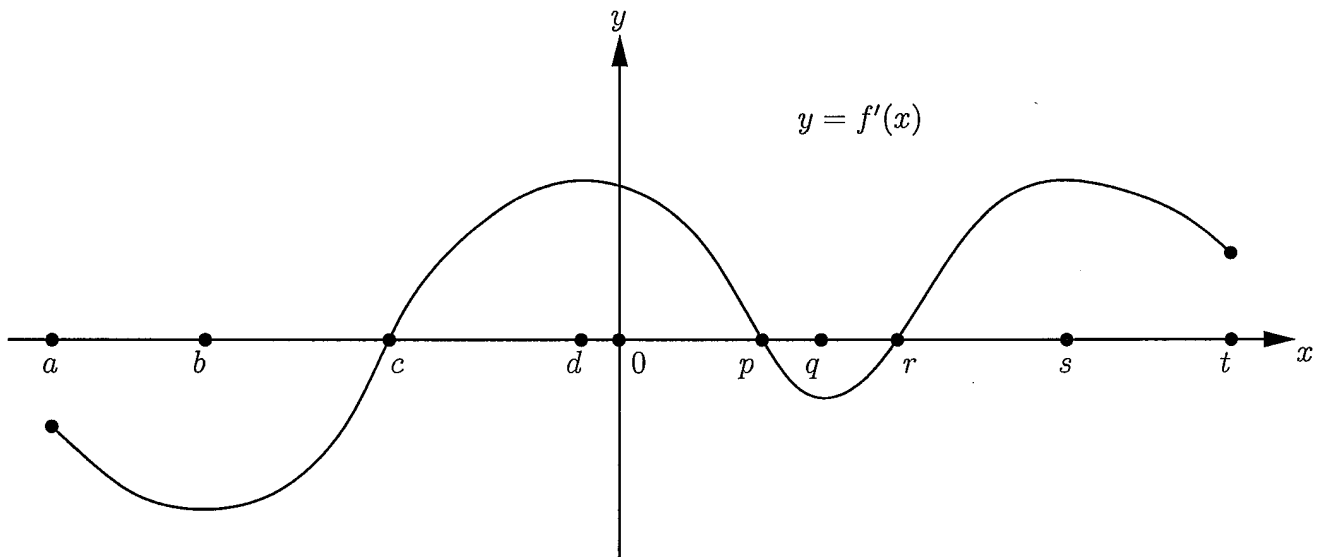
$$y = \frac{3^{\sin x}}{(2x + 1)^{10}}$$

$$\ln y = \sin x \cdot \ln 3 - 10 \ln(2x + 1)$$

$$\frac{1}{y} y' = \cos x \cdot \ln 3 - 10 \frac{1}{2x + 1} \cdot 2$$

$$\therefore y' = \left(\cos x \cdot \ln 3 - \frac{20}{2x + 1} \right) \frac{3^{\sin x}}{(2x + 1)^{10}}$$

Question 6 [8 points]: Consider the graph of $f'(x)$ below (note this is the graph of $f'(x)$, not $f(x)$):



(a) On what interval(s) is f increasing?

$$(c, p) \text{ \& } (r, t)$$

(b) At what value(s) of x does f have local minima?

$$x = c, \quad x = r$$

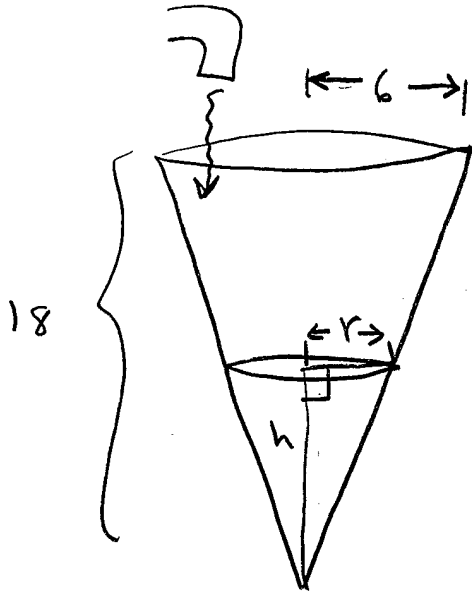
(c) On what interval(s) is the graph of f concave down?

$$(a, b), \quad (d, q), \quad (s, t)$$

(d) At what value(s) of x does the graph of f have points of inflection?

$$x = b, \quad x = d, \quad x = q, \quad x = s$$

Question 7 [10 points]: A water tank has the shape of an inverted circular cone with top radius 6 metres and height 18 metres. Water is pumped into the tank at a rate of 10 cubic metres per minute. How fast is the water level rising when the water is 2 metres deep? (Recall, the volume of a right circular cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)



$$\frac{dV}{dt} = 10 \frac{\text{m}^3}{\text{min.}}$$

Find $\frac{dh}{dt}$ when $h = 2$ m.

$$V = \frac{1}{3} \pi r^2 h.$$

$$\frac{h}{18} = \frac{r}{6}$$

$$\therefore r = \frac{1}{3} h$$

$$\begin{aligned} \therefore V &= \frac{1}{3} \pi \left(\frac{1}{3}h\right)^2 h \\ &= \frac{1}{27} \pi h^3 \end{aligned}$$

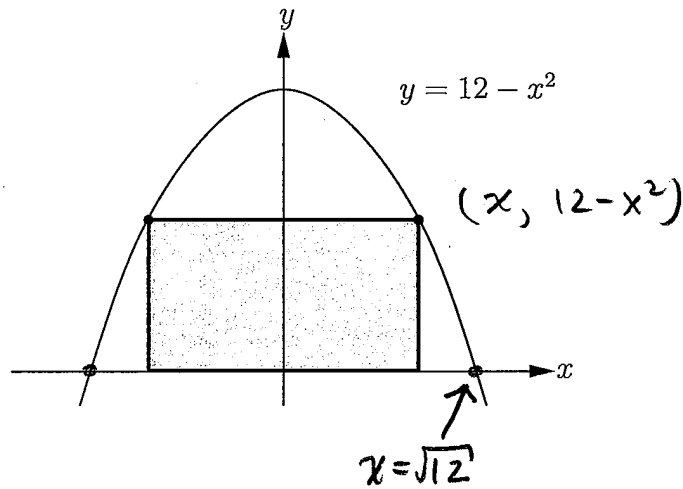
$$\frac{dV}{dt} = \frac{\pi}{27} 3h^2 \frac{dh}{dt}$$

when $h = 2$ m:

$$10 = \frac{\pi}{27} 3 \cdot 2^2 \cdot \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{27 \cdot 10}{3 \cdot 2^2 \pi} = \frac{45}{2\pi} \frac{\text{m}}{\text{min.}}$$

Question 8 [10 points]: A rectangle with base on the x -axis has its upper vertices on the curve $y = 12 - x^2$ (see figure below.) Find the maximum area of such a rectangle.



$$\therefore A(x) = 2x(12 - x^2).$$

Maximize $A(x)$ on $[0, \sqrt{12}]$.

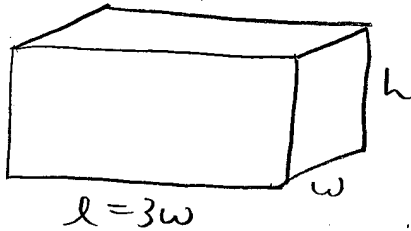
$$\begin{aligned} A'(x) &= 2(12 - x^2) + 2x(-2x) \\ &= 24 - 6x^2 \\ &= 6[4 - x^2] \\ &= 6(2 - x)(2 + x) \end{aligned}$$

$$A'(x) = 0 \quad \text{at} \quad x = 2, -2.$$

x	$A(x)$
0	0
2	$4(12 - 4) = 32$
$\sqrt{12}$	0

\therefore The maximum area is 32 units²,

Question 9 [10 points]: A rectangular box without a top is constructed from cardboard. The length of the box is equal to three times the width. The volume of the box is 18 cubic metres. Find the dimensions of the box which minimize the amount of cardboard used.



Let S = surface area of box.

$$V = 18 \text{ m}^3$$

$$\therefore lwh = 18$$

$$(3w)wh = 18$$

$$3w^2h = 18$$

$$\therefore S' = lw + 2lh + 2wh$$

$$= 3w^2 + 6wh + 2wh$$

$$= 3w^2 + 8wh$$

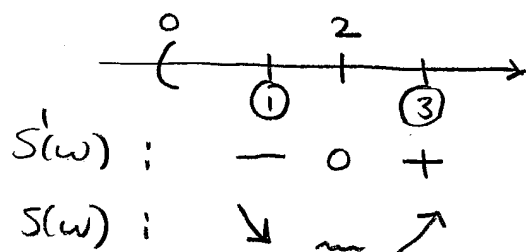
Minimize $S' = 3w^2 + 8wh$ subject to $3w^2h = 18$.

$$3w^2h = 18 \Rightarrow h = \frac{18}{3w^2} = \frac{6}{w^2}$$

$$\therefore S'(w) = 3w^2 + 8w \left(\frac{6}{w^2} \right) = 3w^2 + \frac{48}{w}, \quad w > 0$$

$$S'(w) = 6w - \frac{48}{w^2} = \frac{6w^3 - 48}{w^2}$$

$$S'(w) = 0 \text{ at } w = 2.$$



$\therefore w = 2$ yields the absolute minimum of S' .

$$\therefore w = 2, \quad h = \frac{6}{w^2} = \frac{6}{4} = \frac{3}{2}, \quad l = \frac{18}{wl} = \frac{18}{(2)(\frac{3}{2})} = 6$$

i.e. $w = 2, \quad h = \frac{3}{2} \text{ m}, \quad l = 6 \text{ m}$

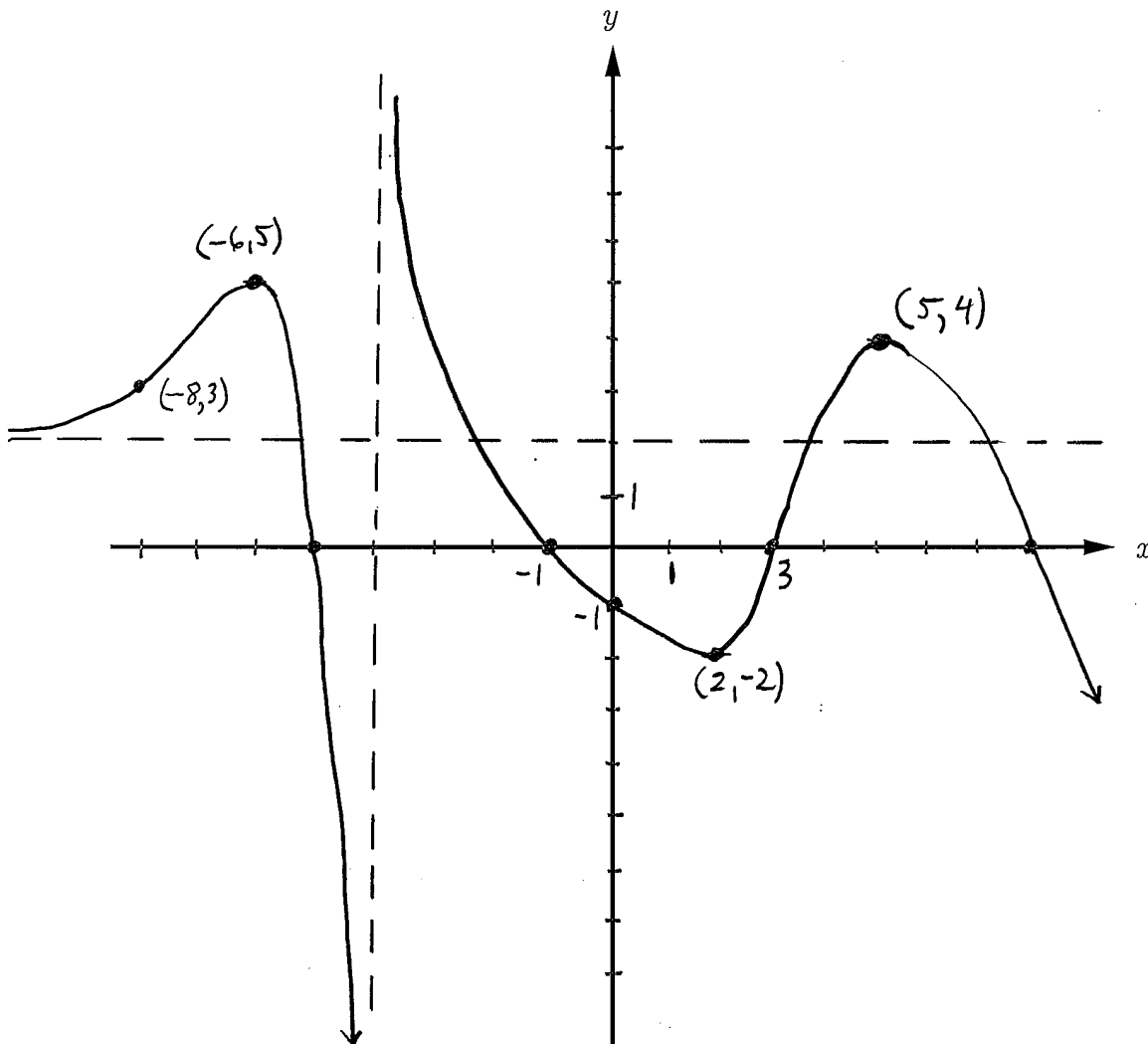
Question 10 [5 points]: Suppose you have analyzed a function and found the following:

- the domain of f is $(-\infty, -4), (-4, \infty)$
- f has the following function values:

x	-8	-6	-5	-1	0	2	3	5	8
$f(x)$	3	5	0	0	-1	-2	0	4	0

- $\lim_{x \rightarrow -\infty} f(x) = 2$, $\lim_{x \rightarrow \infty} f(x) = -\infty$
- $\lim_{x \rightarrow -4^+} f(x) = \infty$, $\lim_{x \rightarrow -4^-} f(x) = -\infty$
- $f'(-6) = f'(2) = f'(5) = 0$
- $f'(x) > 0$ on $(-\infty, -6)$ and $(2, 5)$
- $f'(x) < 0$ on $(-6, -4)$, $(-4, 2)$ and $(5, \infty)$
- $f''(-8) = f''(3) = 0$
- $f''(x) > 0$ on $(-\infty, -8)$ and $(-4, 3)$
- $f''(x) < 0$ on $(-8, -4)$ and $(3, \infty)$

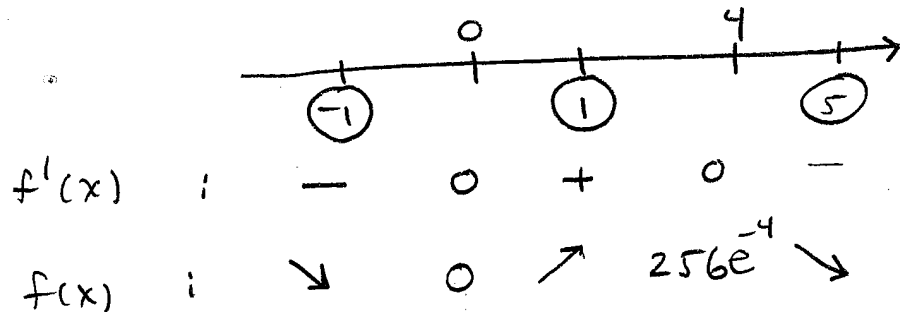
Neatly sketch the graph of $y = f(x)$.



Question 11 [12 points]: The function $f(x) = x^4 e^{-x}$ has first derivative $f'(x) = (4x^3 - x^4)e^{-x}$ and second derivative $f''(x) = (x^4 - 8x^3 + 12x^2)e^{-x}$.

(a) Find the intervals on which $f(x)$ is increasing or decreasing.

$$\begin{aligned} f'(x) = 0 &\Rightarrow 4x^3 - x^4 = 0 \\ &\Rightarrow x^3(4-x) = 0 \\ &\Rightarrow x = 0, 4 \end{aligned}$$



$\therefore f$ increasing on $(0, 4)$, decreasing on $(-\infty, 0) \cup (4, \infty)$.

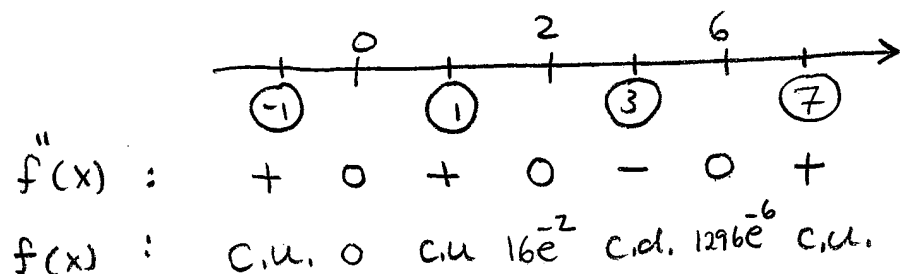
(b) Find the local maximum and minimum values of $f(x)$.

f has a local minimum of 0 at $x = 0$.

f has a local maximum of $256e^{-4}$ at $x = 4$.

(c) Find the intervals on which $f(x)$ is concave up or concave down.

$$\begin{aligned} f''(x) = 0 &\Rightarrow x^4 - 8x^3 + 12x^2 = 0 \\ &x^2(x^2 - 8x + 12) = 0 \\ &x^2(x-2)(x-6) = 0 \\ &x = 0, 2, 6 \end{aligned}$$

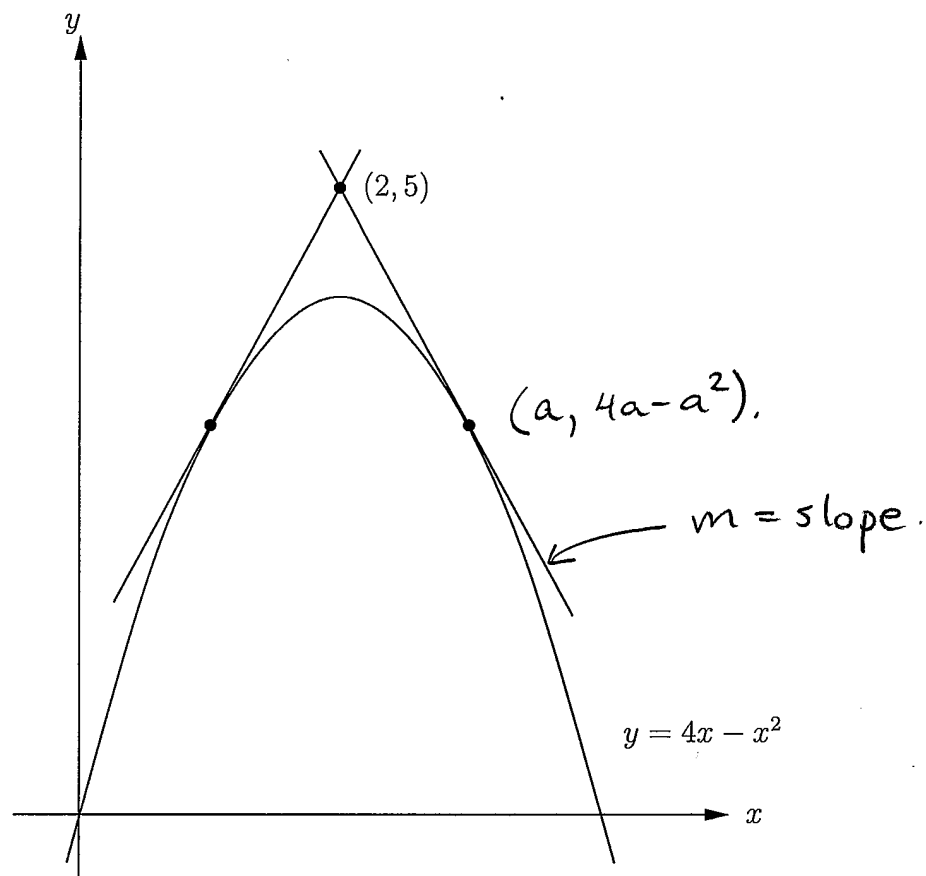


$\therefore f$ is c.u. on $(-\infty, 0) \cup (0, 2) \cup (6, \infty)$; c.d. on $(2, 6)$.

(d) Find the inflection points of $f(x)$.

$$(2, 16e^{-2}), (6, 1296e^{-6})$$

Question 12 [5 points]: There are two tangent lines to the curve $y = 4x - x^2$ that pass through the point $(2, 5)$; see the diagram below. Find the coordinates of the points where these tangent lines intersect the parabola.



$$m = \left. \frac{dy}{dx} \right|_{x=a} = \left. 4 - 2x \right|_{x=a} = 4 - 2a.$$

$$\text{also, } m = \frac{4a - a^2 - 5}{a - 2}$$

$$\therefore \frac{4a - a^2 - 5}{a - 2} = 4 - 2a$$

$$4a - a^2 - 5 = (4 - 2a)(a - 2)$$

$$4a - a^2 - 5 = 4a - 2a^2 - 8 + 4a$$

$$a^2 - 4a + 3 = 0$$

$$(a - 3)(a - 1) = 0$$

$$\therefore a = 1, 3$$

\therefore points are $(1, 3)$ & $(3, 3)$.

$$a = 1 \Rightarrow 4a - a^2 = 3$$

$$a = 3 \Rightarrow 4a - a^2 = 3$$

Question 13 [5 points]: Suppose $f(x) = axe^{bx}$ where a and b are constants. If $f(1/3) = 1$ and $y = f(x)$ has a maximum at $x = 1/3$, find the values of a and b .

$$f\left(\frac{1}{3}\right) = 1 \Rightarrow a\left(\frac{1}{3}\right)e^{\frac{1}{3}b} = 1 \dots \textcircled{1}$$

$$f'\left(\frac{1}{3}\right) = 0 \Rightarrow ae^{bx} + axe^{bx} \cdot b \Big|_{x=\frac{1}{3}} = 0$$

$$\Rightarrow ae^{\frac{1}{3}b} + a \cdot \frac{1}{3} \cdot e^{\frac{1}{3}b} \cdot b = 0$$

$$\Rightarrow ae^{\frac{1}{3}b} \left(1 + \frac{b}{3}\right) = 0 \dots \textcircled{2}$$

$$\therefore a = 0 \quad \text{or} \quad b = -3,$$

but $a \neq 0$ because of $\textcircled{1}$, so $b = -$

$$\therefore \textcircled{1} \Rightarrow a\left(\frac{1}{3}\right)e^{\frac{1}{3}(-3)} = 1$$

$$\frac{a}{3} e^{-1} = 1$$

$$a = 3e$$

$$\boxed{\begin{array}{l} \therefore a = 3e \\ b = -3 \end{array}}$$