Question 1: Evaluate the following limits if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning. (Do not use L'Hospital's rule to evaluate limits.)

(a) $\lim\limits_{x\to -2} \dfrac{2-|x|}{2+x}$ $= \lim\limits_{x\to -2} \dfrac{2-(-x)}{2+x}$ since $x<0$ as $x\to -2$

$= \lim\limits_{x\to -2} \dfrac{2+x}{2+x}$

$= \lim\limits_{x\to -2} 1$

$= \boxed{1}$

[3]

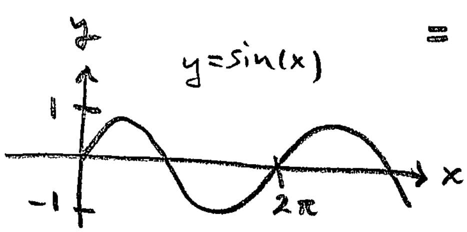
(b) $\lim\limits_{x\to 0} \dfrac{\sin(3x)\sin(5x)}{x^2\cos(x)}$ $= \lim\limits_{x\to 0} \underbrace{\dfrac{\sin(3x)}{3x}}_{\to 1} \cdot \underbrace{\dfrac{\sin(5x)}{5x}}_{\to 1} \cdot \dfrac{(3)(5)}{\cos(x)\}\to 1}$

$= \boxed{15}$

[3]

(c) $\lim\limits_{x\to 2\pi^-} x\csc(x)$ $= \lim\limits_{x\to 2\pi^-} \dfrac{x \;\}\to 2\pi}{\sin(x)\;\}\to 0^-}$

$= \boxed{-\infty}$



[4]

Question 2: Find $\lim_{x\to 0} x^2\cos(\pi/x^5)$.
(Hint: Squeeze Theorem. Be sure to demonstrate that all conditions of the theorem are satisfied, and state a clear conclusion which references use of the theorem.)

$$-x^2 \le x^2\cos\left(\frac{\pi}{x^5}\right) \le x^2$$

Since $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$,

by the Squeeze Thm, $\lim_{x\to 0} x^2\cos\left(\frac{\pi}{x^5}\right) = \boxed{0}$

[5]

Question 3: Let

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Find the constant c that makes f continuous at all real numbers.

For $x \neq 2$, f is defined by polynomials so is continuous. At $x=2$ we must have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-} cx^2+2x = \lim_{x\to 2^+} x^3 - cx = 2^3 - 2c$$

$$\Rightarrow 4c+4 = 8-2c = \cancel{8-2c}$$

$$\Rightarrow 6c = 4$$

$$\Rightarrow \boxed{c = \tfrac{2}{3}}$$

[5]

Question 4: Let $f(x) = \dfrac{1}{x+\sqrt{x}}$. Use the Intermediate Value Theorem to show that $f(x) = 1/3$ for some x in the interval $[1,4]$.

f is continuous on $[1,4]$.

$f(1) = \frac{1}{2}$, $f(4) = \frac{1}{6}$ and $f(4) < \frac{1}{3} < f(1)$.

$\therefore$ By the I.V.T. there is some number c in $(1,4)$ such that $f(c) = \frac{1}{3}$.

[5]

Question 5: Determine $\lim\limits_{t\to\infty} \dfrac{\sqrt{t}+t^2}{2t-t^2}$.

If the limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning. (Do not use L'Hospital's rule to evaluate the limit.)

$$\lim_{t\to\infty} \frac{t^{1/2}+t^2}{2t-t^2} \begin{matrix}\div t^2\\ \div t^2\end{matrix}$$

$$= \lim_{t\to\infty} \frac{t^{-3/2}+1}{2t^{-1}-1}$$

$$= \lim_{t\to\infty} \frac{\left(\frac{1}{t^{3/2}}+1\right)}{\left(\frac{2}{t}-1\right)}$$

$$= \boxed{-1}$$

[5]

Question 6:

(a) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{x}{2-x}$. Neatly show all steps and use proper notation. (No credit will be given if $f'(x)$ is found using derivative rules.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h}{2-x-h} - \frac{x}{2-x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)(2-x) - x(2-x-h)}{(2-x-h)(2-x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{2x} - \cancel{x^2} + 2h - \cancel{hx} - \cancel{2x} + \cancel{x^2} + \cancel{hx}}{(2-x-h)(2-x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{2\cancel{h}}{(2-x-h)(2-x)}\right]$$

$$= \boxed{\frac{2}{(2-x)^2}}$$

[7]

(b) Use your result from part (a) to find the equation of the tangent line to the curve $y = x/(2-x)$ at the point where $x = -1$.

At $x=-1$, $y = \dfrac{-1}{2-(-1)} = \dfrac{-1}{3}$, and slope of tangent line is $y'\big|_{x=-1} = \dfrac{2}{(2-(-1))^2} = \dfrac{2}{9}$. So equation of tangent line is

$$\boxed{y + \frac{1}{3} = \frac{2}{9}(x+1)}$$

[3]

Question 7: Differentiate (that is, find the derivative of) the following functions.

(a) $f(x) = 2x^3 - 3x^2 + 2x + 4$

$$\boxed{f'(x) = 6x^2 - 6x + 2}$$

[2]

(b) $y = 6\sqrt{x} + \dfrac{1}{\sqrt{x}} = 6x^{1/2} + x^{-1/2}$

$$\boxed{y' = 3x^{-1/2} - \tfrac{1}{2}x^{-3/2}}$$

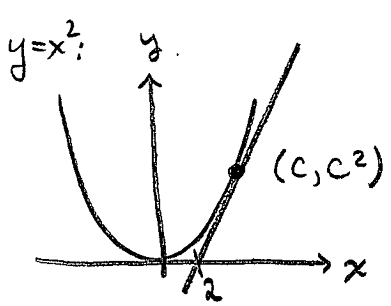
[2]

(c) $g(t) = (3t+1)^2 - \dfrac{\cos(t)}{\pi} = 9t^2 + 6t + 1 - \dfrac{1}{\pi}\cos(t)$

$$\boxed{g'(t) = 18t + 6 + \tfrac{1}{\pi}\sin(t)}$$

[2]

Question 8: Determine the non-zero value of c if the x-intercept of the tangent line to $y = x^2$ at $x = c$ is 2.



Slope of tangent line is $\frac{d}{dx}[x^2]\Big|_{x=c} = 2c$

$\therefore$ Equation of tangent line is

$$y - c^2 = 2c(x-c)$$

Since tangent line has x-intercept 2, $(2,0)$ is on the line, so

$$0 - c^2 = 2c(2-c)$$

$\Rightarrow$ $-c^2 = 4c - 2c^2$

$\Rightarrow$ $c^2 - 4c = 0$

$\Rightarrow$ $c(c-4) = 0$ $\rightarrow$ $\boxed{\cancel{c=0},\ c=4}$

[4]