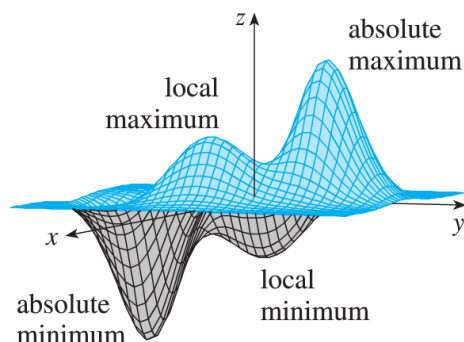


## 11.7: Maximum and Minimum Values:

**Goal:** Use partial derivatives to locate maximum and minimum values of functions of two variables:



### Definition:

- $f$  has a **local maximum** at  $(a, b)$  if  $f(a, b) \geq f(x, y)$  for every  $(x, y)$  in some disk with centre  $(a, b)$ .
- $f$  has a **local minimum** at  $(a, b)$  if  $f(a, b) \leq f(x, y)$  for every  $(x, y)$  in some disk with centre  $(a, b)$ .
- $f$  has an **absolute maximum** at  $(a, b)$  if  $f(a, b) \geq f(x, y)$  for every  $(x, y)$  in the domain of  $f$ .
- $f$  has an **absolute minimum** at  $(a, b)$  if  $f(a, b) \leq f(x, y)$  for every  $(x, y)$  in the domain of  $f$ .

Note: the term *relative minimum* (resp. *maximum*) is equivalent to *local minimum* (resp. *maximum*). The term *global minimum* (resp. *maximum*) is equivalent to *absolute minimum* (resp. *maximum*).

**Theorem:** If  $f$  has a local maximum or minimum at  $(a, b)$  and both  $f_x(a, b)$ ,  $f_y(a, b)$  exist, then  $f_x(a, b) = f_y(a, b) = 0$ .

**Proof:** (in the case of local maximum) If  $f$  has a local maximum at  $(a, b)$  then  $f(x, b)$  has a local maximum at  $x = a$  as a function of one variable, so either  $f'_x(a, b) = 0$  or  $f'_x(a, b)$  does not exist. Since  $f'_x(a, b)$  exists by hypothesis it must be that  $f'_x(a, b) = 0$ . By a similar argument  $f'_y(a, b) = 0$ .

**Definition:** A point  $(a, b)$  is a **critical point** of  $f$  if  $f'_x(a, b) = f'_y(a, b) = 0$  or if at least one of  $f'_x(a, b)$ ,  $f'_y(a, b)$  fails to exist.

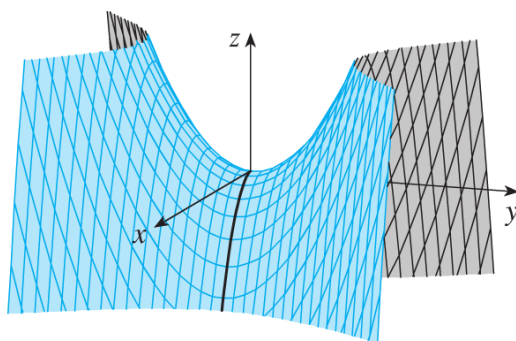
**Conclusion:** Local extrema occur at critical points, but not every critical point corresponds to a local extremum.

As in single variable calculus, the nature of critical points can be determined in part using

**Theorem (Second Derivative Test):** Suppose that  $f_{xx}$ ,  $f_{yy}$ ,  $f_{xy}$  and  $f_{yx}$  are all continuous on a disk with centre  $(a, b)$ , and that  $f_x(a, b) = f_y(a, b) = 0$ . Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If  $D > 0$  and  $f_{xx}(a, b) > 0$  then  $f(a, b)$  is a local minimum.
- If  $D > 0$  and  $f_{xx}(a, b) < 0$  then  $f(a, b)$  is a local maximum.
- If  $D < 0$  then  $f(a, b)$  is neither a local minimum nor maximum;  $(a, b)$  is a *saddle point* (the graph of  $f$  crosses its tangent plane at  $(a, b)$ ):



- If  $D = 0$  then the test fails.

Note:  $D$  above is sometimes called the **Hessian** and the matrix  $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$  the *Hessian matrix*.

**Example:** Determine the local extreme values of  $f(x, y) = (2x - x^2)(2y - y^2)$ .