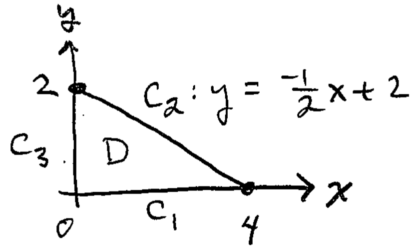
Question 1: Let D be the set of points in the xy-plane that are inside and on the boundary of the triangle with vertices $(0,0)$, $(0,2)$ and $(4,0)$. Find the absolute maximum value of $f(x,y) = x + y - xy$ on D.



Inside D: $f_x(x,y) = 1-y$, $f_y(x,y) = 1-x$ $\Big\}$ $f_x = f_y = 0 \Rightarrow x = y = 1$

$\therefore$ $(1,1)$ is the only c.p. inside D, and $f(1,1) = \boxed{1}$

On boundary of D:

- along C_1 : $y=0$, so $f(x,y) = g_1(x) = x$, $0 \le x \le 4$, which has a maximum of $g_1(4) = \boxed{4}$

- along C_2 : $y = \frac{-1}{2}x + 2$, $0 \le x \le 4$, so

$$f(x,y) = g_2(x) = x + \left(-\tfrac{1}{2}x + 2\right) - x\left(-\tfrac{1}{2}x + 2\right)$$
$$= \frac{x^2}{2} - \frac{3}{2}x + 2$$
$$g_2'(x) = x - \frac{3}{2}$$
$$g_2'(x) = 0 \Rightarrow x = \frac{3}{2}$$

x	$g_2(x)$
0	2
$\frac{3}{2}$	$\frac{7}{8}$
4	4

So $g_2(x)$ has an absolute maximum of $\boxed{4}$

- along C_3 : $x = 0$, so $f(x,y) = g_3(y) = y$, $0 \le y \le 2$ which has a maximum of $g_3(2) = \boxed{2}$.

$\therefore$ f has an absolute maximum of 4 at $(4,0)$.

[10]

Question 2: Use the method of Lagrange multipliers to find both the maximum and minimum values of $f(x, y, z) = x + y - z$ over the sphere $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 1$.

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x,y,z) = 1 \end{array}\right\} \Rightarrow \begin{array}{ll} ① & 1 = 2\lambda x \\ ② & 1 = 2\lambda y \\ ③ & -1 = 2\lambda z \\ ④ & x^2+y^2+z^2=1 \end{array}$$

①, ②, ③ $\Rightarrow$ $x \neq 0, y \neq 0, z \neq 0, \lambda \neq 0$.

① ÷ ②: $\frac{1}{1} = \frac{2\lambda x}{2\lambda y} \Rightarrow x = y$

① ÷ ③: $\frac{1}{-1} = \frac{2\lambda x}{2\lambda z} \Rightarrow z = -x$

④ $\Rightarrow$ $x^2 + (x)^2 + (-x)^2 = 1$

$\Rightarrow 3x^2 = 1$

$\Rightarrow x^2 = \frac{1}{3}$

$$\Rightarrow \begin{cases} x = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}}, \ z = \frac{-1}{\sqrt{3}} \\ x = \frac{-1}{\sqrt{3}} \Rightarrow y = \frac{-1}{\sqrt{3}}, \ z = \frac{1}{\sqrt{3}} \end{cases}$$

$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = 3\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3}$

$f\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 3\left(\frac{-1}{\sqrt{3}}\right) = -\sqrt{3}$

∴ f has an abs. max. of $\sqrt{3}$ and an abs. min. of $-\sqrt{3}$ on $x^2+y^2+z^2=1$

[10]

Question 3: Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

$$V = \iint_R (3y^2 - x^2 + 2)\, dA$$

$$= \int_{-1}^{1} \int_{1}^{2} (3y^2 - x^2 + 2)\, dy\, dx$$

$$= \int_{-1}^{1} \left[y^3 - x^2 y + 2y \right]_1^2 dx$$

$$= \int_{-1}^{1} (8 - 2x^2 + 4) - (1 - x^2 + 2)\, dx$$

$$= \int_{-1}^{1} -x^2 + 9\, dx$$

$$= 2\int_0^1 9 - x^2\, dx$$

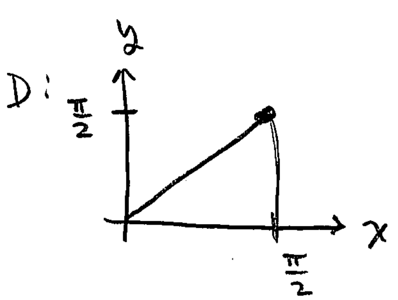
$$= 2\left[9x - \frac{x^3}{3} \right]_0^1$$

$$= 2\left[9 - \frac{1}{3} \right]$$

$$= \boxed{\frac{52}{3}}$$

[5]

Question 4: Evaluate the following double integral:



$$\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin x}{x}\, dx\, dy$$

$$= \int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_{x=0}^{\pi/2} \frac{\sin x}{x} \left[y \right]_0^x dx$$

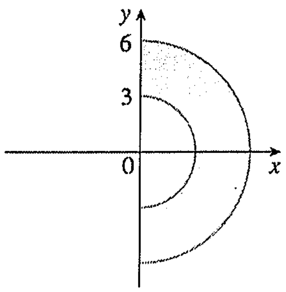
$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cancel{x}} \cancel{[x - 0]}\, dx$$

$$= \left[-\cos(x) \right]_0^{\frac{\pi}{2}} = -\overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}} - (-\cos(0))$$

$$= \boxed{1}$$

[5]

Question 5: Evaluate $\iint_D \frac{y}{x^2+y^2}\,dA$ where D is the shaded region in the following figure:



In polar coordinates:

$$I = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=3}^{6} \left(\frac{r\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \right) r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{3}^{6} \frac{\cancel{r^2}\sin\theta}{\cancel{r^2}(\underbrace{\cos^2\theta+\sin^2\theta}_{1})}\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,[r]_3^6\,d\theta$$

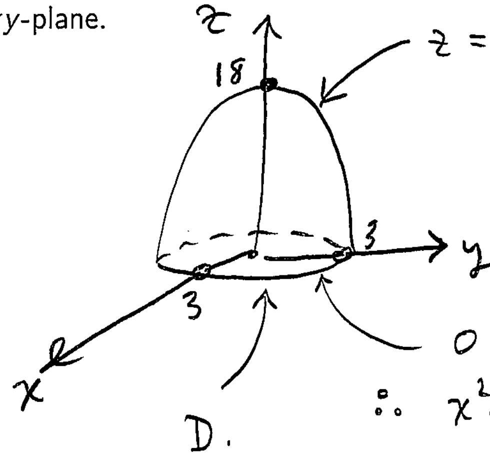
$$= \int_{-\pi/2}^{\pi/2} \sin\theta\,[6-3]\,d\theta$$

$$= 3[-\cos(\theta)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \boxed{0}$$

[5]

Question 6: Determine the volume of the region below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane.



$z = 18 - 2x^2 - 2y^2$

$0 = 18 - 2x^2 - 2y^2$

$\therefore\ x^2 + y^2 = 9$

$$\therefore\ V = \iint_D (18 - 2x^2 - 2y^2)\,dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} (18 - 2r^2\cos^2\theta - 2r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 18r - 2r^3\,dr$$

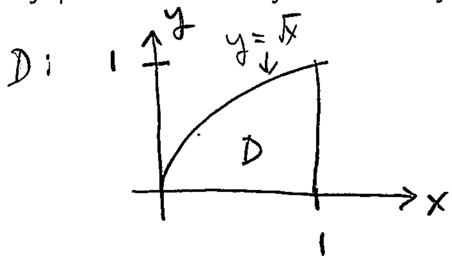
$$= 2\pi \left[9r^2 - \frac{1}{2}r^4\right]_0^3$$

$$= 2\pi\left[81 - \frac{81}{2}\right]$$

$$= \boxed{81\pi}$$

[5]

Question 7: Compute $\iiint_E 6xy\,dV$ where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$.



$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{x}}\int_{z=0}^{1+x+y} 6xy \, dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{x}} 6xy\left[z\right]_0^{1+x+y} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{x}} \left(6xy + 6x^2y + 6xy^2\right) dy\,dx$$

$$= 6\int_{x=0}^{1} \left[\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_0^{\sqrt{x}} dx$$

$$= 6\int_0^1 \left(\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^{5/2}}{3}\right) dx$$

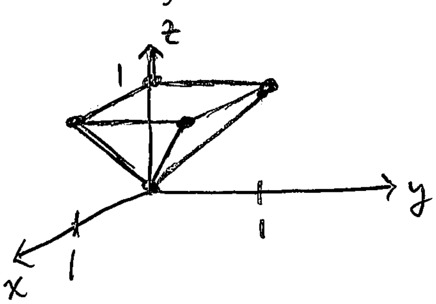
$$= 6\left[\frac{x^3}{6} + \frac{x^4}{8} + \frac{2x^{7/2}}{21}\right]_0^1$$

$$= 6\left(\frac{1}{6} + \frac{1}{8} + \frac{2}{21}\right) = \boxed{\frac{65}{28}}$$

[7]

Question 8: Fill in the boxes with the appropriate bounds of integration:

$$\int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy = \int_{\boxed{0}}^{\boxed{1}}\int_{\boxed{0}}^{\boxed{z}}\int_{\boxed{0}}^{\boxed{z}} f(x,y,z)\,dy\,dx\,dz$$



[3]