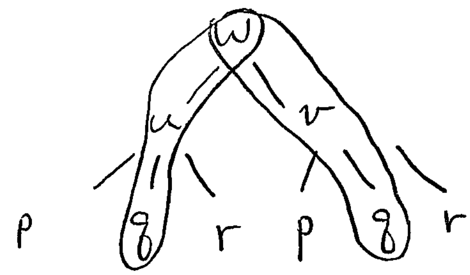
Question 1: Let

$$W = \frac{v}{2u+v}$$

where

$$u = pq\sqrt{r} \quad \text{and} \quad v = p\sqrt{q}r$$

Determine $\frac{\partial W}{\partial q}$ when $p = 2$, $q = 1$ and $r = 4$.



$$\frac{\partial W}{\partial q} = \frac{\partial W}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial W}{\partial v}\frac{\partial v}{\partial q}$$

$$= \frac{-2v}{(2u+v)^2} \cdot p\sqrt{r} + \frac{2u+\cancel{v}-\cancel{v}}{(2u+v)^2} \cdot \frac{pr}{2\sqrt{q}}$$

At $(p,q,r) = (2,1,4)$, $u = (2)(1)\sqrt{4} = 4$ & $v = (2)\sqrt{1}(4) = 8$

$$\therefore \frac{\partial W}{\partial q} = \frac{-2(8)}{(2(4)+8)^2} \cdot (2)\sqrt{4} + \frac{2(4)}{(2(4)+8)^2} \cdot \frac{\cancel{(2)}(4)}{\cancel{2}\sqrt{1}} = \frac{-1}{4} + \frac{1}{8} = \boxed{-\frac{1}{8}}$$

[5]

Question 2: Let $T(x,y)$ be the temperature in degrees Celcius at point (x,y), and suppose a bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$ and $y = 2 + \frac{1}{3}t$ where x and y are measured in centimetres. If $T_x(2,3) = 4$ and $T_y(2,3) = 3$, how fast is the temperature rising along the bug's path after 3 seconds? State units with your answer.

$$\left.\frac{dT}{dt}\right|_{t=3} = \left.\frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt}\right|_{t=3}$$

$$= \left.(4)\frac{1}{2\sqrt{1+t}} + (3)\left(\frac{1}{3}\right)\right|_{t=3}$$

$$= 1 + 1$$

$$= \boxed{2 \text{ degrees/second}}$$

[5]

Question 3: Find the directional derivative of $f(x, y) = x^2e^{-y}$ at the point $(-2, 0)$ in the direction toward the point $(2, -3)$) .

Here $\vec{u} = \dfrac{\langle 2-(-2), -3-0\rangle}{|\langle 2-(-2), -3-0\rangle|} = \dfrac{\langle 4, -3\rangle}{\sqrt{4^2+(-3)^2}} = \langle \frac{4}{5}, -\frac{3}{5}\rangle$

$\nabla f(-2,0) = \langle 2xe^{-y}, -x^2e^{-y}\rangle\Big|_{(-2,0)} = \langle -4, -4\rangle$

$\therefore\ D_{\vec{u}} f(-2,0) = \nabla f(-2,0) \cdot \vec{u}$

$= \langle -4, -4\rangle \cdot \langle \frac{4}{5}, \frac{-3}{5}\rangle$

$= \boxed{\frac{-4}{5}}$

[5]

Question 4: Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point $(2, 1)$ and the direction in which it occurs.

$\nabla f(2,1) = \langle 2xy, x^2 + \frac{1}{2\sqrt{y}}\rangle\Big|_{(2,1)}$

$= \langle 2(2)(1), 2^2 + \frac{1}{2\sqrt{1}}\rangle$

$= \langle 4, \frac{9}{2}\rangle$

$\therefore$ Maximum rate of change is $|\nabla f(2,1)| = \sqrt{4^2 + (\frac{9}{2})^2} = \boxed{\frac{\sqrt{145}}{2}}$,

occurs in direction of $\nabla f = \boxed{\langle 4, \frac{9}{2}\rangle}$

[5]

Question 5: Find the equation of the tangent plane to the surface

$$x^4 + y^4 + z^4 = 3x^2y^2z^2$$

at the point $(1, 1, 1)$.

$F(x,y,z) = x^4+y^4+z^4-3x^2y^2z^2 = 0$

$F_x(1,1,1) = 4x^3 - 6xy^2z^2\Big|_{(1,1,1)} = -2 \underbrace{= F_y(1,1,1) = F_z(1,1,1)}_{\text{by symmetry}}$.

$\therefore \nabla F(1,1,1) = \langle -2,-2,-2\rangle$

$\therefore$ Equation of tangent plane is

$\nabla F(1,1,1)\cdot\langle x-1, y-1, z-1\rangle = 0 \Rightarrow \boxed{-2(x-1)-2(y-1)-2(z-1)=0 \text{ or } x+y+z=3}$ **[5]**

Question 6: At what point(s) does the normal line through $(1, 2, 1)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 12$ intersect the sphere $x^2 + y^2 + z^2 = 102$?

Normal line has direction $\nabla F(1,2,1) = \langle 8x, 2y, 8z\rangle\Big|_{(1,2,1)} = \langle 8,4,8\rangle$

$\nabla F(1,2,1) = 4\langle 2,1,2\rangle$, so use $\langle 2,1,2\rangle$ as direction vector for line.

$\therefore$ Normal line is $\vec{r}(t) = \langle 1,2,1\rangle + t\langle 2,1,2\rangle = \langle 1+2t, 2+t, 1+2t\rangle$.

Normal line intersects $x^2+y^2+z^2 = 102$

when $(1+2t)^2 + (2+t)^2 + (1+2t)^2 = 102$

$\Rightarrow 1+4t+4t^2+4+4t+t^2+1+4t+4t^2 = 102$

$\Rightarrow 9t^2+12t-96 = 0$

$\Rightarrow 3[3t^2+4t-32] = 0$

$\Rightarrow 3[3t^2-4t+8t-32] = 0$

$\Rightarrow 3[(3t-8)(t+4)] = 0$

$\therefore t = \frac{8}{3},\ t = -4$

$\therefore$ points of intersection are $\vec{r}\left(\frac{8}{3}\right) = \left\langle \frac{19}{3}, \frac{14}{3}, \frac{19}{3}\right\rangle$, $\vec{r}(-4) = \langle -7, -2, -7\rangle$

[5]

Question 7: Find all critical points of $f(x, y) = (x^2 + y^2)e^{-x}$ and classify each as either a local maximum, local minimum or a saddle point. Carefully calculate all required derivatives and keep your work organized.

$$f_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x} = -e^{-x}\left[x^2-2x+y^2\right].$$

$$f_y(x,y) = 2ye^{-x}.$$

$$f_y = 0 \Rightarrow y=0$$

For $y=0$, $f_x = 0 \Rightarrow -e^{-x}\left[x^2-2x+0^2\right]=0$

$$\Rightarrow x^2-2x=0$$

$$\Rightarrow x(x-2)=0$$

$$\Rightarrow x=0, \quad x=2.$$

$\therefore$ C.P.'s are $(0,0)$ & $(2,0)$.

$$f_{xx} = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + (x^2+y^2)e^{-x} = e^{-x}\left[x^2+y^2-4x+2\right]$$

$$f_{yy} = 2e^{-x}$$

$$f_{xy} = f_{yx} = -2ye^{-x}.$$

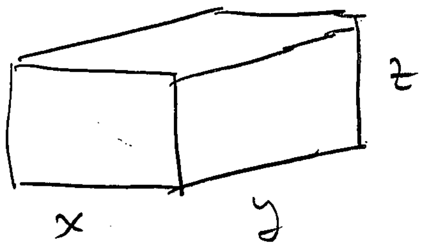
$$\therefore D = f_{xx}f_{yy} - (f_{xy})^2 = 2e^{-2x}\left[x^2+y^2-4x+2\right] - \left[-2ye^{-x}\right]^2$$

$$= 2e^{-2x}\left[x^2-y^2-4x+2\right].$$

C.P.	$D = 2e^{-2x}\left[x^2-y^2-4x+2\right]$	f_{xx}	Conclusion
$(0,0)$	$4>0$	2	local min.
$(2,0)$	$-4e^{-4}<0$	—	saddle pt.

[10]

Question 8: Find the dimensions of a rectangular box (that is, a conventional box with six rectangular faces) having maximum volume if the sum of the lengths of the 12 edges is a constant c. (The edges of the box are the llines where the rectangular faces meet).



Maximize $V = xyz$

subject to $L = 4x + 4y + 4z = c$,

$x > 0,\ y > 0,\ z > 0$.

$$z = \frac{c - 4x - 4y}{4} = \frac{c}{4} - x - y.$$

$$\therefore V = xy\left(\frac{c}{4} - x - y\right) = \frac{c}{4}xy - x^2y - xy^2$$

$$V_x = \frac{c}{4}y - 2xy - y^2 = y\left(\frac{c}{4} - 2x - y\right) = 0 \quad ①$$

$$V_y = \frac{c}{4}x - 2xy - x^2 = x\left(\frac{c}{4} - 2y - x\right) = 0 \quad ②.$$

From ① : $y = \frac{c}{4} - 2x$, (since $y > 0$)

From ② : $y = \frac{c}{8} - \frac{x}{2}$, (since $x > 0$),

$$\therefore \frac{c}{4} - 2x = \frac{c}{8} - \frac{x}{2}$$

$$\Rightarrow \frac{3}{2}x = \frac{c}{8}$$

$$\Rightarrow x = \left(\frac{2}{3}\right)\left(\frac{c}{8}\right) = \frac{c}{12}$$

$$\therefore y = \frac{c}{4} - 2\left(\frac{c}{12}\right) = \frac{c}{12}$$

$$\therefore z = \frac{c}{12}$$

Note: On physical grounds we know an abs. max. of V exists and this solution must occur at a C.P. We found a single C.P and so this must correspond to the abs. max. of V.

$\boxed{\therefore \text{Dimensions are } x = y = z = \frac{c}{12}.}$

[10]