Question 1: Find the unit tangent vector to $\mathbf{r}(t) = \langle e^{-t}\cos(t), e^{-t}\sin(t), e^{-t}\rangle$ at the point $(1, 0, 1)$.

$(1,0,1)$ corresponds to $t=0$.

$$\vec{r}\,'(0) = \langle -e^{-t}\cos(t) - e^{-t}\sin(t),\ -e^{-t}\sin(t) + e^{-t}\cos(t),\ -e^{-t}\rangle \Big|_{t=0}$$

$$= \langle -1, 1, -1\rangle$$

$$|\vec{r}\,'(0)| = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$\therefore$ unit tangent is $\vec{T} = \dfrac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \boxed{\left\langle \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle}$

[5]

Question 2: For this question recall Newton's Second Law

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

where **F** is force in Newtons (N), m is mass in kilograms (kg) and $\mathbf{a}$ is acceleration in m/s^2.
A force with magnitude 10 N acts directly upward from the xy-plane on an object of mass 5 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Find the position function $\mathbf{r}(t)$ for the object.

$\vec{F}(t) = m\vec{a}(t) = 10\hat{k}$

$\Rightarrow 5\vec{a}(t) = 10\hat{k}$

$\Rightarrow \vec{a}(t) = \left(\frac{10}{5}\right)\hat{k}$

$\Rightarrow \vec{r}\,''(t) = 2\hat{k}$,

$\vec{r}(0) = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$,

$\vec{r}\,'(0) = \hat{\imath} - \hat{\jmath}$.

$\therefore \vec{r}\,'(t) = 2t\hat{k} + C_1$

$\vec{r}\,'(0) = \hat{\imath} - \hat{\jmath} \Rightarrow C_1 = \hat{\imath} - \hat{\jmath}$

$\therefore \vec{r}\,'(t) = \hat{\imath} - \hat{\jmath} + 2t\hat{k}$

$\therefore \vec{r}(t) = t\hat{\imath} - t\hat{\jmath} + t^2\hat{k} + C_2$

$\vec{r}(0) = \vec{0} \Rightarrow C_2 = \vec{0}$

$\boxed{\therefore \vec{r}(t) = t\hat{\imath} - t\hat{\jmath} + t^2\hat{k}}$

[5]

Question 3: Compute the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy}$$

$$= \lim_{(x,y)\to(0,0)} \frac{[(x+y)-(x-y)][(x+y)+(x-y)]}{xy}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(2y)(2x)}{(xy)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{4\cancel{xy}}{\cancel{xy}}$$

$$= \boxed{4}$$

[5]

Question 4: Show that the following limit does not exist (carefully explain your reasoning):

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

- Letting $(x,y) \to (0,0)$ along the x-axis (so $y=0$):

$$\frac{xy^3}{x^2+y^6} = \frac{0}{x^2+0} \to 0 \text{ as } x \to 0$$

- Letting $(x,y) \to (0,0)$ along curve $x=y^3$:

$$\frac{x\cdot y^3}{x^2+y^6} = \frac{y^3 y^3}{(y^3)^2+y^6} = \frac{1}{2} \text{ as } x \to 0.$$

Since different approach paths of (x,y) to $(0,0)$ yield different limiting values of $\frac{xy^3}{x^2+y^6}$, the limit does not exist.

[5]

Question 5:

(i) Let $f(x,y,z) = \ln(1+e^{xyz})$. Calculate $f_y(2,0,-1)$

$$f_y = \frac{1}{1+e^{xyz}} \cdot e^{xyz} \cdot xz$$

$$\therefore f_y(2,0,-1) = \frac{1}{1+e^0} \cdot e^0 \cdot 2 \cdot (-1) = \frac{1}{\cancel{2}} \cdot (1) \cdot (\cancel{2}) \cdot (-1)$$

$$= \boxed{-1}$$

[2]

(ii) Let $w = x\sin(xy-z)$. Calculate $\left[\frac{\partial w}{\partial x} - \frac{\partial w}{\partial z}\right]_{(1,\pi/2,0)}$

$$\left[\frac{\partial w}{\partial x} - \frac{\partial w}{\partial z}\right]_{(1,\frac{\pi}{2},0)} = \left[1\cdot\sin(xy-z) + x\cos(xy-z)\cdot y + x\cos(xy-z)\right]_{(1,\frac{\pi}{2},0)}$$

$$= \left[\sin\left(\tfrac{\pi}{2}\right) + \cancel{\cos\left(\tfrac{\pi}{2}\right)\left(\tfrac{\pi}{2}\right)}^{0} + \cancel{\cos\left(\tfrac{\pi}{2}\right)}^{0}\right]$$

$$= \boxed{1}$$

[2]

(iii) Use implicit differentiation to find $\partial z/\partial y$: $yz + x\ln y = z^2$

$$\frac{\partial}{\partial y}\left[yz + x\ln y\right] = \frac{\partial}{\partial y}\left[z^2\right]$$

$$(1)z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y}\left[y - 2z\right] = -\frac{x}{y} - z$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-\left(\frac{x}{y}+z\right)}{y-2z}}$$

[2]

Question 6: Determine if the function $u(x,y) = x^3 + 3xy^2$ is a solution to Laplace's equation $u_{xx} + u_{yy} = 0$.

$u_x = 3x^2 + 3y^2$

$u_{xx} = 6x$

$u_y = 6xy$

$u_{yy} = 6x$

$u_{xx} + u_{yy} = 6x + 6x = 12x \neq 0$

so $\boxed{\text{No!}}$

[4]

Question 7: Find an equation of the tangent plane to the graph of $f(x, y) = \cos(x/y)$ at the point $(\pi, 4)$.

$$f_x(\pi,4) = -\sin\left(\frac{x}{y}\right)\cdot\frac{1}{y}\Big|_{(\pi,4)} = -\sin\left(\frac{\pi}{4}\right)\cdot\frac{1}{4} = \frac{-1}{4\sqrt{2}}$$

$$f_y(\pi,4) = -\sin\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right)\Big|_{(\pi,4)} = \frac{\pi}{16\sqrt{2}}$$

$$f(\pi,4) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

$$\therefore\quad z = f(\pi,4) + f_x(\pi,4)(x-\pi) + f_y(\pi,4)(y-4)$$

$$\boxed{z = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x-\pi) + \frac{\pi}{16\sqrt{2}}(y-4)}$$

[5]

Question 8: Determine all points (x, y, z) at which tangent planes to the surface $z = xye^{-(x^2+y^2)/2}$ are horizontal.

Tangent planes horizontal

$\Rightarrow$ Equation of tangent plane has form $z = k$, a constant

$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$ at such points.

$$\frac{\partial z}{\partial x} = ye^{-\frac{(x^2+y^2)}{2}} + xy\,e^{-\frac{(x^2+y^2)}{2}}\left(-\frac{\cancel{2}x}{\cancel{2}}\right) = y(1-x^2)e^{-\left(\frac{x^2+y^2}{2}\right)}.$$

$\therefore \frac{\partial z}{\partial x} = 0 \Rightarrow y = 0$ or $x = \pm 1$.

Similarly, $\frac{\partial z}{\partial y} = x(1-y^2)e^{-\left(\frac{x^2+y^2}{2}\right)} = 0$ at $x = 0$ or $y = \pm 1$

$\therefore$ Tangent plane is horizontal at $\begin{cases}(0,0,0),\ (1,-1,e^{-1}),\ (-1,-1,e^{-1}) \\ (1,1,e^{-1}),\ (-1,1,e^{-1})\end{cases}$

[5]

Question 9: Use a linear approximation to estimate $f(0.1, -0.05)$ where $f(x, y) = \sqrt{y + \cos^2 x}$.

$$L(x,y) = f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y.$$

$$= \sqrt{0+\cos^2(0)} + \frac{1}{2}\left[y+\cos^2(x)\right]^{-\frac{1}{2}} \cdot \cancel{2\cos(x)\cdot(-\sin(x))}\Big|_{(0,0)} \cdot x \quad \to 0$$

$$+ \frac{1}{2}\left[y+\cos^2(x)\right]^{-\frac{1}{2}} \cdot 1 \Big|_{(0,0)} \cdot y$$

$$= 1 + \frac{1}{2}\left[0+\cos^2(0)\right]^{-\frac{1}{2}} y$$

$$= 1 + \frac{1}{2}y.$$

$$\therefore f(0.1, -0.05) \approx L(0.1, -0.05) = 1 + \frac{1}{2}\left(\frac{-1}{20}\right) = \boxed{\frac{39}{40}}$$

[5]

Question 10: Recall that the volume of a right circular cone of base radius r and height h is $V = \pi r^2 h/3$. Suppose a cone has equal base radius and height, and that the height is then reduced by a small amount k. Use differentials (or linear approximation) to determine the approximate amount by which the radius must change so that the total volume of the cone remains unchanged.

$$V = \frac{\pi r^2 h}{3}, \quad dh = -k$$

Find dr so that $dV = 0$:

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$

$$dV = \frac{2\pi}{3}rh\,dr + \frac{\pi r^2}{3}dh$$

$$\therefore 0 = \frac{2\pi}{3}rh\,dr + \frac{\pi r^2}{3}(-k)$$

$$\left.\begin{array}{l} 0 = \frac{2\pi}{3}r^2 dr + \frac{\pi}{3}r^2(-k) \end{array}\right\} \text{ since } r = h$$

$$0 = \frac{\pi r^2}{3}\left[2dr - k\right]$$

$$\therefore 2dr - k = 0$$

$$\Rightarrow dr = \frac{k}{2}.$$

$\therefore$ radius must <u>increase</u> by approximately $\boxed{\frac{k}{2}}$.

[5]