**Question 1:**

**(a)** Find all points on the sphere of radius 1 and centre (0,0,0) that are a distance $1/\sqrt{2}$ from each of the $xy$-plane and $xz$-plane.

Let $(x,y,z)$ be such a point.

Distance to $xy$-plane $= \frac{1}{\sqrt{2}} \Rightarrow z = \pm\frac{1}{\sqrt{2}}$

Distance to $xz$-plane $= \frac{1}{\sqrt{2}} \Rightarrow y = \pm\frac{1}{\sqrt{2}}$

Since $(x,y,z)$ is on sphere of radius 1,

$$x^2+y^2+z^2=1$$

$$\Rightarrow x^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=1$$

$$\Rightarrow x^2+\frac{1}{2}+\frac{1}{2}=1$$

$$\Rightarrow x=0$$

$\therefore$ points are $\boxed{\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right),\left(0,\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)}$ **[5]**

**(b)** A sphere with radius 5 and centre (2, −6, 4) intersects the $xy$ plane to form a circle. Determine the radius of this circle.

Sphere has equation $(x-2)^2+(y+6)^2+(z-4)^2=5^2$.

On $xy$-plane $z=0$, so circle has equation

$$(x-2)^2+(y+6)^2+(0-4)^2=5^2$$

$$(x-2)^2+(y+6)^2=25-16$$

$$(x-2)^2+(y+6)^2=3^2$$

$\boxed{\therefore \text{ radius of circle is } 3}$

**[5]**

**Question 2:** For this question use the vectors

$$\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{b} = 3\mathbf{i} - \mathbf{k}$$

**(a)** Compute $|2\mathbf{a} - 4\mathbf{b}|$. $= |2\langle 1,-3,2\rangle - 4\langle 3,0,-1\rangle|$

$$= |\langle -10,-6,8\rangle|$$

$$= \sqrt{(-10)^2+(-6)^2+8^2}$$

$$= \boxed{10\sqrt{2}}$$

**[3]**

**(b)** Find a vector $\mathbf{c}$ which when added to $\mathbf{a}+\mathbf{b}$ gives $\mathbf{a}-\mathbf{b}$.

$$\vec{c} = (\vec{a}-\vec{b}) - (\vec{a}+\vec{b})$$

$$= -2\vec{b}$$

$$= -2\langle 3,0,-1\rangle$$

$$= \boxed{\langle -6,0,2\rangle \text{ or } -6\hat{\imath}+2\hat{k}}$$

**[2]**

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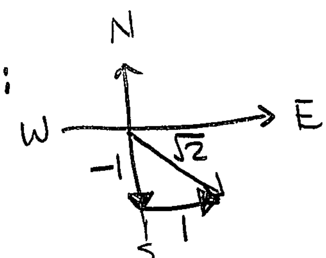
**Question 3:** A ship is travelling north at 20 km/hr. A woman on the deck of the ship walks south-east at $\sqrt{2}$ km/hr. What is the speed of the woman relative to the surface of the water?

<u>Ship</u>: $\vec{v}_s = 20\hat{\jmath}\ \frac{km}{hr}$.

<u>Woman</u>:




$\vec{v}_w = \hat{\imath} - \hat{\jmath}\ \frac{km}{hr}$.

$\therefore\ \vec{v} = \vec{v}_S + \vec{v}_W = \hat{\imath} + 19\hat{\jmath}$

$\therefore$ Speed is

$$|\vec{v}| = \sqrt{1^2+19^2}$$

$$= \boxed{\sqrt{362}\ \frac{km}{hr}}$$

$$\doteq 19\ \frac{km}{hr}.$$

**[5]**

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**Question 4:** Determine all angles in the triangle with vertices $P(1,-3,-2)$, $Q(2,0,-4)$ and $R(6,-2,-5)$. (If giving a calculator answer round your final answers to one decimal place.)


$$\vec{PQ} \bullet \vec{PR} = |\vec{PQ}|\,|\vec{PR}|\cos(\angle_{QPR})$$

$$\langle 1,3,-2\rangle \circ \langle 5,1,-3\rangle = \sqrt{14}\sqrt{35}\cos(\angle_{QPR})$$

$$14 = \sqrt{14}\sqrt{35}\cos(\angle_{QRR})$$

$$\therefore \angle_{QPR} = \arccos\left(\frac{14}{\sqrt{14}\sqrt{35}}\right) = \arccos\left(\sqrt{\frac{2}{5}}\right) \doteq \boxed{50.8^\circ}$$

$$\vec{QP} \bullet \vec{QR} = |\vec{QP}|\,|\vec{QR}|\cos(\angle_{PQR})$$

$$\langle -1,-3,2\rangle \bullet \langle 4,-2,-1\rangle = \sqrt{15}\sqrt{21}\cos(\angle_{PQR})$$

$$0 = \sqrt{15}\sqrt{21}\cos(\angle_{PQR})$$

$$\therefore \boxed{\angle_{PQR} = 90^\circ}$$

$$\boxed{\therefore \angle_{QRP} \doteq 90-50.8 = 39.2^\circ}$$

**[5]**

**Question 5:** What angle does the vector $\langle 1,2,3\rangle$ make with the $xy$-plane? (If giving a calculator answer round your final answers to one decimal place.)

Projection of $\langle 1,2,3\rangle$ onto $xy$-plane is $\langle 1,2,0\rangle$.

Angle $\theta$ of interest is that between $\langle 1,2,3\rangle$ & $\langle 1,2,0\rangle$:

$$\langle 1,2,3\rangle \circ \langle 1,2,0\rangle = |\langle 1,2,3\rangle|\,|\langle 1,2,0\rangle|\cos\theta$$

$$5 = \sqrt{14}\sqrt{5}\cos\theta$$

$$\therefore \cos\theta = \frac{5}{\sqrt{14}\sqrt{5}} = \sqrt{\frac{5}{14}}$$

$$\boxed{\therefore \theta = \arccos\left(\sqrt{\frac{5}{14}}\right) \doteq 53.3^\circ}$$

**[5]**

**Question 6:** Determine the area of the triangle with vertices $P(1,-3,-2)$, $Q(2,0,-4)$ and $R(6,-2,-5)$.

$$A = \frac{1}{2}\left|\vec{PQ} \times \vec{PR}\right|$$

$$= \frac{1}{2}\left|\langle 1,3,-2\rangle \times \langle 5,1,-3\rangle\right|$$

$$= \frac{1}{2}\left|\det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}\right|$$

$$= \frac{1}{2}\left|-7\hat{i} - 7\hat{j} - 14\hat{k}\right|$$

$$= \frac{\sqrt{(-7)^2 + (-7)^2 + (-14)^2}}{2}$$

$$= \frac{\sqrt{294}}{2} = \boxed{\frac{7\sqrt{6}}{2}}$$

[5]

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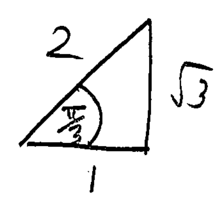
**Question 7:** If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1,2,2\rangle$ find the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\vec{a} \cdot \vec{b} = \sqrt{3} \Rightarrow |\vec{a}|\,|\vec{b}| \cos\theta = \sqrt{3} \quad (1)$$

$$\vec{a} \times \vec{b} = \langle 1,2,2\rangle \Rightarrow |\vec{a}|\,|\vec{b}| \sin\theta = |\langle 1,2,2\rangle| = 3 \quad (2)$$

$$(2) \div (1) \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{3}{\sqrt{3}} = \sqrt{3}$$




$$\therefore \tan\theta = \sqrt{3}$$

$$\boxed{\therefore \quad \theta = \frac{\pi}{3} = 60^\circ}$$

[5]

**Question 8:** Find an equation of the line through $A(1, 0, -2)$ which is orthogonal to the plane containing the points $P(1, -3, -2)$, $Q(2, 0, -4)$ and $R(6, -2, -5)$ . State your answer in parametric form.

$$\vec{PQ} \times \vec{PR}$$
$$= \langle 1,3,-2\rangle \times \langle 5,1,-3\rangle$$
$$= \langle -7,-7,-14\rangle \text{ using Question 6}$$
$$= -7\langle 1,1,2\rangle.$$

$\therefore$ Using $\langle 1,1,2\rangle$ for direction vector of line and $A(1,0,-2)$ as point on the line:

$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 1,0,-2\rangle + t\langle 1,1,2\rangle$.

In parametric form: $\boxed{\begin{aligned} x &= 1+t \\ y &= t \\ z &= -2+2t \end{aligned}}$

[5]

**Question 9:** Find an equation of the plane through $P(1, -1, 1)$ that is parallel to both $\mathbf{r}_1 = \langle 2, 1, 3\rangle + t\langle 2, 1, 3\rangle$ and $\mathbf{r}_2 = \langle 1, 2, -5\rangle + t\langle 1, 1, 1\rangle$ .

Normal to plane is $\vec{n} = \langle 2,1,3\rangle \times \langle 1,1,1\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$
$$= -2\hat{i} + \hat{j} + \hat{k}$$
$$= \langle -2,1,1\rangle.$$

$\therefore$ using $P(1,-1,1)$ and $\vec{n} = \langle -2,1,1\rangle$:

$$(\langle x,y,z\rangle - \langle 1,-1,1\rangle)\cdot\langle -2,1,1\rangle = 0$$
$$-2(x-1) + (y+1) + (z-1) = 0$$
$$\boxed{-2x+y+z = -2}$$

[5]