Question 1:

(a) Use the trapezoid rule on 4 subintervals to approximate $\int_0^{2\pi} 4\sqrt{1+\cos(x)}\,dx$. Simplify your final answer.

Here $\Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}$

x_0	x_1	x_2	x_3	x_4
0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

$f(x) = 4\sqrt{1+\cos(x)}$

$$\therefore \int_0^{2\pi} 4\sqrt{1+\cos(x)}\,dx \approx T_4 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{(\pi/2)}{2}\left[4\sqrt{2} + (2)(4) + (2)(0) + (2)(4) + 4\sqrt{2}\right]$$

$$= \frac{\pi}{4}\left[16 + 8\sqrt{2}\right]$$

$$= \boxed{2\pi\left[2+\sqrt{2}\right]}.$$

[5]

(b) It can be shown that $f(x) = 4\sqrt{1+\cos(x)}$ has second derivative $f''(x) = -\sqrt{1+\cos(x)}$ and fourth derivative $f^{(4)}(x) = \frac{\sqrt{1+\cos(x)}}{4}$. Use this information to give an error bound on your approximation in part (a). Simplify your final answer.

$$|E_{T_4}| \leq \frac{K(b-a)^3}{12n^2} \quad \text{where } b = 2\pi,\ a = 0,\ n = 4$$

and $K \leq \max_{[0,2\pi]} |f''(x)|$

$$= \max_{[0,2\pi]} \left|-\sqrt{1+\cos(x)}\right|$$

$$= \sqrt{2}$$

$$\therefore |E_{T_4}| \leq \frac{\sqrt{2}(2\pi - 0)^3}{12 \cdot 4^2}$$

$$= \frac{\sqrt{2} \cdot \cancel{2^3} \cdot \pi^3}{3 \cdot \cancel{2^6}\, 2^3} = \boxed{\frac{\sqrt{2}\,\pi^3}{24}}$$

[5]

Question 2: Evaluate the following improper integral making proper use of any required limits:

$$\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^3}}\, dx$$

$$= \lim_{b \to 1^-} \int_{-1}^{b} \frac{x^2}{\sqrt{1-x^3}}\, dx \qquad \left.\begin{array}{l} u = 1 - x^3 \\ du = -3x^2 dx \end{array}\right\}$$

$$= \lim_{b \to 1^-} \left[-\frac{2}{3}\sqrt{1-x^3} \right]_{-1}^{b}$$

$$= \lim_{b \to 1^-} -\frac{2}{3}\left(\sqrt{1-b^3} - \sqrt{1-(-1)^3} \right)$$

$$= \boxed{\frac{2\sqrt{2}}{3}}$$

[5]

Question 3: Use the comparison theorem to show that the following integral converges:

$$I = \int_{1}^{\infty} \frac{2+\sin(x)}{\sqrt{x}(1+x)}\, dx$$

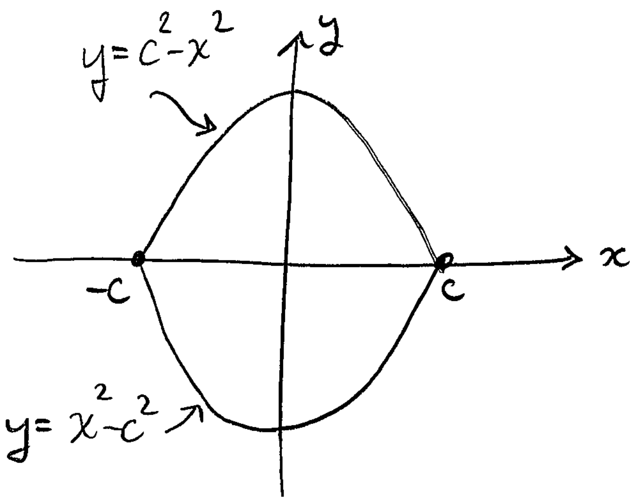
$$0 \leq \frac{2+\sin(x)}{\sqrt{x}\,(1+x)} \leq \frac{3}{x^{3/2}} .$$

Since $\int_1^\infty \frac{3}{x^{3/2}}\, dx = 3\int_1^\infty \frac{1}{x^{3/2}}\, dx$ converges,

so does I by the comparison theorem.

[5]

Question 4: Find the value of c so that the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ has area 72.



By symmetry, we require

$$\int_0^c (c^2 - x^2)\,dx = \frac{72}{4} = 18$$

$$\therefore \left[c^2 x - \frac{x^3}{3}\right]_0^c = 18$$

$$c^3 - \frac{c^3}{3} = 18$$

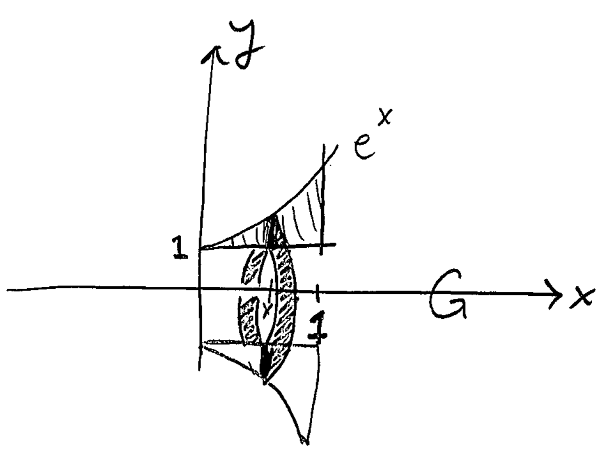
$$\frac{2}{3}c^3 = 18$$

$$c^3 = \frac{(18)(3)}{2}$$

$$\boxed{c = 3}$$

[5]

Question 5: The region in the first quadrant that is bounded by the curves $y = e^x$, $y = 1$ and $x = 1$ is rotated about the x-axis. Determine the volume of the resulting solid.



$$V = \int_0^1 \pi\left[(e^x)^2 - 1^2\right]dx$$

$$= \pi \int_0^1 e^{2x} - 1 \; dx$$

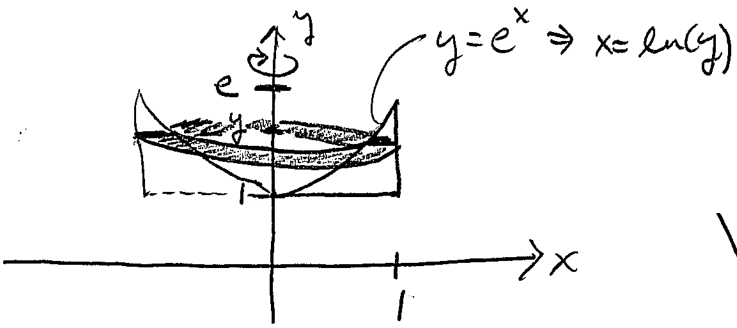
$$= \pi \left[\frac{e^{2x}}{2} - x\right]_0^1$$

$$= \pi\left[\left(\frac{e^2}{2} - 1\right) - \left(\frac{e^0}{2} - 0\right)\right]$$

$$= \boxed{\frac{\pi(e^2 - 3)}{2}}$$

[5]

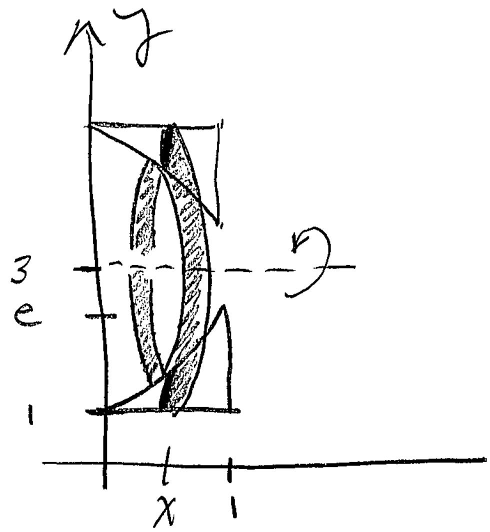
Question 6: The same region as in Question 5 (the region in the first quadrant bounded by $y = e^x$, $y = 1$ and $x = 1$) is rotated about the y-axis. Find an integral representing the volume of the resulting solid. DO NOT EVALUATE THE INTEGRAL, simply set it up.



$$V = \int_{y=1}^{e} \pi\left[1^2 - [\ln(y)]^2\right] dy$$

[2]

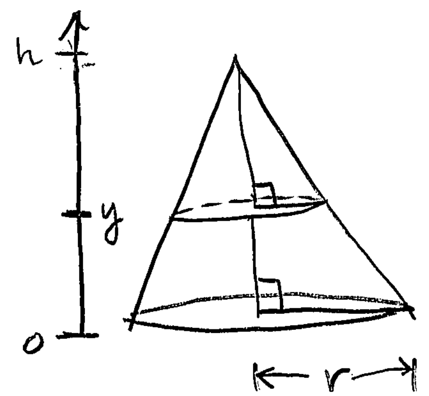
Question 7: The same region as in Question 5 (the region in the first quadrant bounded by $y = e^x$, $y = 1$ and $x = 1$) is rotated about the horizontal line $y = 3$. Find an integral representing the volume of the resulting solid. DO NOT EVALUATE THE INTEGRAL, simply set it up.

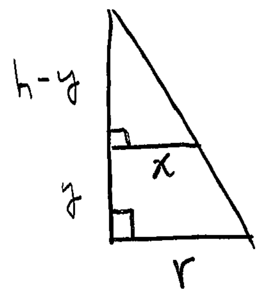


$$V = \int_0^1 \pi\left[2^2 - (3-e^x)^2\right] dx$$

[3]

Question 8: Determine the volume of a cone of base radius r and height h .





By similar Δ's:

$$\therefore \frac{x}{h-y} = \frac{r}{h}$$

$$\therefore x = \frac{r}{h}(h-y).$$

$$\therefore A(y) = \pi x^2 = \pi\left[\frac{r}{h}(h-y)\right]^2$$

$$\therefore V = \int_0^h A(y)\,dy$$

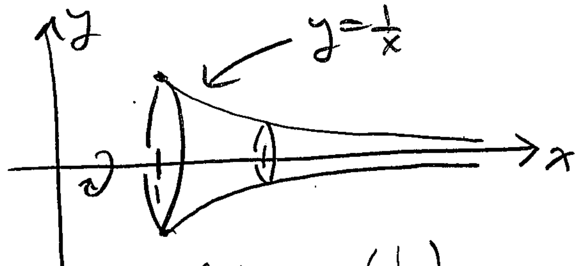
$$= \int_0^h \frac{\pi r^2}{h^2}(h-y)^2\,dy$$

$$= \frac{\pi r^2}{h^2}\left[\frac{(h-y)^3}{-3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2}\cdot\frac{h^3}{3} = \boxed{\frac{1}{3}\pi r^2 h}$$

[5]

Question 9: The region between the curve $y = 1/x$ and the x-axis over the interval $[1, \infty)$ is rotated about the x-axis. Determine the volume of the infinitely long solid that is generated.



$$A(x) = \pi\left(\frac{1}{x^2}\right).$$

$$\therefore V = \int_1^\infty \pi\left(\frac{1}{x^2}\right)dx$$

$$= \lim_{b\to\infty}\int_1^b \pi\left(\frac{1}{x^2}\right)dx$$

$$= \lim_{b\to\infty} \pi\left[\frac{-1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \pi\left[-\frac{1}{b} - \left(\frac{-1}{1}\right)\right] = \boxed{\pi}$$

[5]