Question 1: (Substitution Method)

(a) Determine $\frac{1}{2}\int 2e^{2x} \sin\left(e^{2x}\right) dx.$ $= I$

Let $u = e^{2x}$

$du = 2e^{2x}\, dx$

$\therefore I = \frac{1}{2} \int \sin(u)\, du$

$= -\frac{1}{2} \cos(u) + C$

$= \boxed{-\frac{1}{2} \cos(e^{2x}) + C}$

[3]

(b) Determine $\displaystyle\int \frac{\cos(x)}{1+\sin^2(x)}\, dx.$ $= I$

Let $u = \sin x$

$du = \cos x\, dx$

$\therefore I = \displaystyle\int \frac{1}{1+u^2}\, du$

$= \arctan(u) + C$

$= \boxed{\arctan(\sin x) + C}$

[3]

(c) Evaluate $\displaystyle\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\, dx.$ $= I$

Let $u = \ln x$

$du = \frac{1}{x}\, dx$

$x = e \Rightarrow u = \ln e = 1$

$x = e^2 \Rightarrow u = \ln(e^2) = 2$

$\therefore I = \displaystyle\int_1^2 \frac{1}{\sqrt{u}}\, du$

$= 2\left[\sqrt{u}\right]_1^2 = \boxed{2(\sqrt{2}-1)}$

[4]

Question 2: (Integration by Parts)

(a) Evaluate $\int_0^1 (x^2+1)e^x\,dx$. $= I$

$u = x^2+1 \quad dv = e^x dx$

$du = 2x\,dx \quad v = e^x$

$\therefore I = \int_0^1 u\,dv = [uv]_0^1 - \int_0^1 v\,du$

$= \left[(x^2+1)e^x\right]_0^1 - \underbrace{\int_0^1 2xe^x\,dx}_{u=2x \quad dv=e^x dx,\ du=2dx \quad v=e^x}$

$= 2e-1 - \left[[2xe^x]_0^1 - \int_0^1 2e^x\,dx\right]$

$= 2e-1 - \left[2e - 0 - [2e^x]_0^1\right]$

$= \cancel{2e} - 1 - \cancel{2e} + 2e - 2$

$= \boxed{2e-3}$

[5]

(b) Determine $\int \sin(\ln x)\,dx$. $= I$

$u = \sin(\ln x) \quad dv = dx$

$du = \frac{\cos(\ln x)}{x}dx \quad v = x$

$\therefore I = \int u\,dv = uv - \int v\,du$

$= x\sin(\ln x) - \underbrace{\int \cancel{x}\frac{\cos(\ln x)}{\cancel{x}}\,dx}_{u=\cos(\ln x) \quad dv=dx,\ du=-\frac{\sin(\ln x)}{x} \quad v=x}$

$= x\sin(\ln x) - \left[x\cos(\ln x) - \int \cancel{x}\left(-\frac{\sin(\ln x)}{\cancel{x}}\right)dx\right]$

$\therefore I = x\sin(\ln x) - x\cos(\ln x) - I$

$\therefore I = \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C$

[5]

Question 3: (Trigonometric Integrals)

(a) Evaluate $\int_0^{2\pi} \cos^2(\theta/4)\, d\theta$.

$\Big\}$ Recall: $\cos^2 u = \dfrac{1+\cos(2u)}{2}$

$$= \int_0^{2\pi} \frac{1+\cos\left(\cancel{2}\frac{\theta}{\cancel{4}2}\right)}{2}\, d\theta$$

$$= \left[\frac{\theta}{2} + \sin\left(\theta/2\right)\right]_0^{2\pi}$$

$$= \left(\frac{2\pi}{2} + \cancel{\sin\left(\frac{2\pi}{2}\right)}^{0}\right) - \left(\cancel{\frac{0}{2} + \sin(0)}^{0}\right)$$

$$= \boxed{\pi}$$

[5]

(b) Determine $\int \tan^5(x)\sec^3(x)\, dx$. $= I$

$$I = \int \tan^4(x)\sec^2(x)\sec(x)\tan(x)\, dx$$

$$= \int (\sec^2(x)-1)^2\sec^2(x)\sec(x)\tan(x)\, dx$$

Let $u = \sec(x)$

$du = \sec(x)\tan(x)\, dx$

$$\therefore\ I = \int (u^2-1)^2 u^2\, du$$

$$= \int u^6 - 2u^4 + u^2\, du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sec^7(x)}{7} - \frac{2\sec^5(x)}{5} + \frac{\sec^3(x)}{3} + C}$$

[5]

Question 4: (Trigonometric Substitution) Determine

$$I = \int \frac{1}{x^2\sqrt{x^2+9}}\,dx$$

Let $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$$\therefore\ I = \int \frac{3\sec^2\theta}{9\tan^2\theta\sqrt{9\tan^2\theta+9}}\,d\theta$$

$$= \frac{3}{9}\int \frac{\sec^{\cancel{2}}\theta}{\tan^2\theta \cdot 3 \cdot \cancel{\sec\theta}}\,d\theta$$

$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta$$

Let $u = \sin\theta$

$du = \cos\theta\, d\theta$

$$I = \frac{1}{9}\int u^{-2}\,du$$

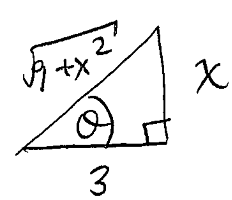
$$= \frac{-1}{9}\,\frac{1}{u} + C$$

$$= \frac{-1}{9}\,\frac{1}{\sin\theta} + C$$

$$= \boxed{-\frac{1}{9}\,\frac{\sqrt{9+x^2}}{x} + C}$$

$x = 3\tan\theta$

$\therefore\ \tan\theta = \frac{x}{3}$



$\therefore\ \sin\theta = \frac{x}{\sqrt{9+x^2}}$

[10]

Question 5: (Partial Fractions) Determine

$$\int \frac{x^2+3x+2}{x(x^2+1)}\,dx$$

$$\frac{x^2+3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{(A+B)x^2 + Cx + A}{x(x^2+1)}$$

$\therefore\ A+B=1$

$C=3$

$A=2$

$\therefore\ B = 1-A = -1$

$$\therefore\ I = \int \frac{2}{x} + \frac{-x+3}{x^2+1}\,dx$$

$$= \int \frac{2}{x} - \frac{1}{2}\underbrace{\int \frac{2x}{x^2+1}\,dx}_{\substack{u=x^2+1 \\ du=2x\,dx}} + 3\int \frac{1}{x^2+1}\,dx$$

$$= \boxed{2\ln|x| - \frac{1}{2}\ln|x^2+1| + 3\arctan(x) + C}$$

[10]