Question 1: A particle starting at initial position $s(0) = 0$ m and initial velocity of k m/s has acceleration at time t second of $a(t) = k\sqrt{t}$ m/s^2. Here k is a constant. Find the particle's position after 1 second.

$a(t) = kt^{1/2}$

$\therefore v(t) = \frac{2}{3}kt^{3/2} + C_1$

$v(0) = k \Rightarrow C_1 = k$

$\therefore v(t) = \frac{2}{3}kt^{3/2} + k$

$\therefore s(t) = \frac{4}{15}kt^{5/2} + kt + C_2$

$s(0) = 0 \Rightarrow C_2 = 0$

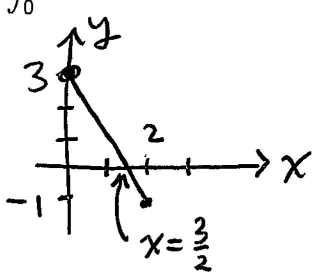
$\therefore s(t) = \frac{4}{15}kt^{5/2} + kt$

$\therefore s(1) = \frac{4}{15}k + k = \boxed{\frac{19}{15}k \text{ m}}$

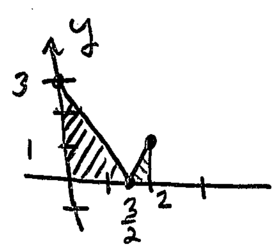
[5]

Question 2: Use an area interpretation to find $\int_0^2 |3 - 2x|\, dx$

$y = 3 - 2x$ has graph



so $y = |3 - 2x|$ has graph



$\therefore \int_0^2 |3 - 2x|\, dx$

$=$ shaded area

$= (\frac{1}{2})(\frac{3}{2})(3) + (\frac{1}{2})(\frac{1}{2})(1) = \boxed{\frac{5}{2}}$

[5]

Question 3: Use the definition of the definite integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

to evaluate

$$\int_{-2}^{1}(1-2x^2)\,dx$$

Carefully set up the Riemann sum and clearly show the steps of your simplification.

$[a,b] = [-2,1]$

$f(x) = 1-2x^2$

$\Delta x = \frac{b-a}{n} = \frac{1-(-2)}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = -2 + i\left(\frac{3}{n}\right)$

$$f(x_i) = 1-2x_i^2 = 1-2\left[-2+i\left(\tfrac{3}{n}\right)\right]^2$$

$$= 1-2\left[4 - \frac{12i}{n} + \frac{9i^2}{n^2}\right]$$

$$= -7 + \frac{24}{n}i - \frac{18}{n^2}i^2$$

$$\int_{-2}^{1}(1-2x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[-7+\frac{24}{n}i-\frac{18}{n^2}i^2\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[-\frac{21}{n}+\frac{72}{n^2}i-\frac{54}{n^3}i^2\right]$$

$$= \lim_{n\to\infty}\left[\left(\sum_{i=1}^{n}-\frac{21}{n}\right)+\left(\frac{72}{n^2}\right)\left(\sum_{i=1}^{n} i\right)-\left(\frac{54}{n^3}\right)\left(\sum_{i=1}^{n} i^2\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(-\frac{21}{\cancel{n}}\right)(\cancel{n}) + \frac{\cancel{72}^{36}}{\cancel{2}_1}\cdot\frac{\cancel{n}}{\cancel{n}}\cdot\frac{(n+1)}{n} - \frac{\cancel{54}^{9}}{\cancel{6}_1}\cdot\frac{\cancel{n}}{\cancel{n}}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right]$$

$$= -21 + 36 - (9)(2)$$

$$= \boxed{-3}$$

Check!

$$\int_{-2}^{1}(1-2x^2)\,dx$$

$$= \left[x - \frac{2}{3}x^3\right]_{-2}^{1}$$

$$= \left(1-\frac{2}{3}\right) - \left(-2 - \frac{2}{3}(-8)\right)$$

$$= \frac{1}{3} - \frac{10}{3}$$

$$= \frac{-9}{3}$$

$$= -3$$ ☺

[10]

Question 4: Find the average value of $f(x) = \pi(\sin(x) + \sec^2(x)$ over the interval $[-\pi/4, \pi/4]$. (State your answer as a single fraction.)

$$f_{ave} = \frac{1}{(\frac{\pi}{4})-(\frac{-\pi}{4})} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi\left(\sin(x) + \sec^2(x)\right) dx$$

$$= \left(\frac{2}{\cancel{\pi}}\right)(\cancel{\pi}) \left[-\cos(x) + \tan(x)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2\left[\left(-\frac{1}{\sqrt{2}} + 1\right) - \left(-\frac{1}{\sqrt{2}} - 1\right)\right]$$

$$= \boxed{4}$$

[5]

Question 5: A certain population is currently of size 20 individuals and is increasing at a rate of $r(t) = (1+t^2)/3$ individuals per week. How large will the population be in three weeks time?

$$P = 20 + \int_0^3 \frac{1+t^2}{3}\, dt$$

$$= 20 + \frac{1}{3}\left[t + \frac{t^3}{3}\right]_0^3$$

$$= 20 + \frac{1}{3}\left[\left(3 + \frac{3^3}{3}\right) - 0\right]$$

$$= \boxed{24} \text{ individuals.}$$

[5]

Question 6:

(a) Evaluate $\int_4^9 \sqrt{x} - \frac{1}{\sqrt{x}}\, dx$. $= \int_4^9 x^{1/2} - x^{-1/2}\, dx$

$$= \left[\frac{2}{3} x^{3/2} - 2x^{1/2}\right]_4^9$$

$$= \left[\frac{2}{3}(9^{3/2}) - 2(9^{1/2})\right] - \left[\frac{2}{3}(4^{3/2}) - 2(4^{1/2})\right]$$

$$= (18 - 6) - \left(\frac{16}{3} - 4\right)$$

$$= 12 - \frac{4}{3} = \boxed{\frac{32}{3}}$$

[3]

(b) Evaluate $\int_0^1 x^{10} + 10^x\, dx$.

$$= \left[\frac{x^{11}}{11} + \frac{10^x}{\ln(10)}\right]_0^1$$

$$= \left(\frac{1}{11} + \frac{10}{\ln(10)}\right) - \left(0 + \frac{1}{\ln(10)}\right)$$

$$= \boxed{\frac{1}{11} + \frac{9}{\ln(10)}}$$

[3]

Question 7: Suppose f is a continuous function with the property that

$$\int_0^x f(t)\, dt = \sin(2x) - \int_0^x \cos(2t) f(t)\, dt\ .$$

Find a formula for $f(x)$.

$$\frac{d}{dx}\int_0^x f(t)\, dt = \frac{d}{dx}\left[\sin(2x) - \int_0^x \cos(2t) f(t)\, dt\right]$$

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$$f(x) = 2\cos(2x) - \cos(2x) f(x)$$

$$\therefore\ f(x)(1 + \cos(2x)) = 2\cos(2x)$$

$$\boxed{f(x) = \frac{2\cos(2x)}{1 + \cos(2x)}}$$

[4]

Question 8: (Substitution Method)

(a) Determine $\frac{1}{8}\int \frac{8x}{(4x^2+1)^5}\,dx$.

$u = 4x^2+1$
$du = 8x\,dx$

$$= \frac{1}{8}\int u^{-5}\,du$$

$$= \frac{1}{8}\frac{u^{-4}}{-4} + C$$

$$= \boxed{\frac{-1}{32}\frac{1}{(4x^2+1)^4} + C}$$

[3]

(b) Evaluate $\frac{1}{\pi}\int_1^{\sqrt{e}} \frac{\pi\sin(\pi \ln x)}{x}\,dx$.

$u = \pi \ln x$
$du = \frac{\pi}{x}\,dx$

$x = 1 \Rightarrow u = 0$
$x = \sqrt{e} \Rightarrow u = \pi \ln(e^{1/2}) = \frac{\pi}{2}$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin(u)\,du$$

$$= \frac{1}{\pi}\left[-\cos(u)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[-0-(-1)\right] = \boxed{\frac{1}{\pi}}$$

[3]

(c) Determine $2\int \frac{e^{\sec(\sqrt{x})}\sec(\sqrt{x})\tan(\sqrt{x})}{2\sqrt{x}}\,dx$.

$u = \sec(\sqrt{x})$
$du = \sec(\sqrt{x})\tan(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)dx$

$$= 2\int e^u\,du$$

$$= 2e^u + C$$

$$= \boxed{2e^{\sec(\sqrt{x})} + C}$$

[4]