Question 1:

(a) Find the linear approximation $T_1(x)$ about $a = 2$ for $f(x) = \frac{1}{x+2}$.

$f(x) = \frac{1}{x+2}$; $f(2) = \frac{1}{2+2} = \frac{1}{4}$

$f'(x) = \frac{-1}{(x+2)^2}$; $f'(2) = \frac{-1}{(2+2)^2} = \frac{-1}{16}$

$\therefore T_1(x) = f(a) + f'(a)(x-a)$

$= \boxed{\frac{1}{4} - \frac{1}{16}(x-2)}$

[4]

(b) Use your result in part (a) to approximate $f(3/2)$. (Express your answer as a single simplified fraction.)

$f\left(\frac{3}{2}\right) \approx T_1\left(\frac{3}{2}\right) = \frac{1}{4} - \frac{1}{16}\left(\frac{3}{2} - 2\right)$

$= \frac{1}{4} - \frac{1}{16}\left(-\frac{1}{2}\right)$

$= \frac{8}{32} + \frac{1}{32}$

$= \boxed{\frac{9}{32}}$

[2]

(c) Give an error bound for your approximation in part (b). (Again, express your answer as a single simplified fraction.)

$R_1(x) = \frac{1}{2} f''(z)(x-a)^2$ where $f''(z) = \frac{2}{(z+2)^3}$, $a = 2$, $x = \frac{3}{2}$

and $\frac{3}{2} < z < 2$

$\therefore |R_1(x)| = \left| \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{(z+2)^3} \cdot \left(3/2 - 2\right)^2 \right|$

$= \left| \frac{1}{(z+2)^3} \cdot \frac{1}{4} \right|$

$\leq \left| \frac{1}{\left(\frac{3}{2}+2\right)^3} \cdot \frac{1}{4} \right| = \frac{2^3}{7^3} \cdot \frac{1}{4} = \boxed{\frac{2}{7^3}}$

[4]

Question 2:

(a) Use a Taylor (or Maclaurin) polynomial of degree 3 to approximate $e^{-0.1}$. (Express your answer as a single simplified fraction.)

Here $f(x) = e^x$, $a = 0$.

$T_3(x)$ consists of the terms up to degree 3 of the Taylor series for e^x:

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

$$\therefore e^{-0.1} \approx T_3\left(-\tfrac{1}{10}\right) = 1 - \frac{1}{10} + \frac{(-1/10)^2}{2} + \frac{(-1/10)^3}{6}$$

$$= 1 - \frac{1}{10} + \frac{1}{200} - \frac{1}{6000}$$

$$= \frac{6000 - 600 + 30 - 1}{6000}$$

$$= \boxed{\frac{5429}{6000}}$$

[5]

(b) Give an error bound on your approximation in part (a). (Again, express your answer as a single simplified fraction.)

$$R_3(x) = \frac{1}{4!} f^{(4)}(z)\, x^4 \quad \text{where } f^{(4)}(z) = e^z,\ a = 0,\ x = -\tfrac{1}{10};$$
$$-\tfrac{1}{10} < z < 0.$$

$$\therefore \left|R_3\left(-\tfrac{1}{10}\right)\right| = \left|\frac{1}{4!} e^z \cdot \left(\frac{-1}{10}\right)^4\right|$$

$$\le \frac{1}{24} \cdot e^0 \cdot \frac{1}{10^4}$$

$$= \boxed{\frac{1}{240\,000}}$$

[5]

Question 3:

(a) Find the first four nonzero terms of the Maclaurin series for $f(x) = \dfrac{\sin(x)}{1+x}$.

$$\sin(x) \cdot \frac{1}{1+x}$$

$$= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right]\left[1 - x + x^2 - x^3 + x^4 - \dots\right]$$

$$= x - x^2 + \left(1 - \frac{1}{3!}\right)x^3 + \left(-1 + \frac{1}{3!}\right)x^4 + [\text{terms of degree 5 and higher}]$$

$$= \boxed{x - x^2 + \frac{5}{6}x^3 - \frac{5}{6}x^4 + \dots}$$

[4]

(b) Determine $f^{(4)}(0)$, the fourth derivative of f at 0, where $f(x)$ is as in part (a). There is no need to simplify factorials (if any) in your final answer.

$$\frac{f^{(4)}(0)\,x^4}{4!} = \frac{-5}{6}x^4$$

$$\therefore\ f^{(4)}(0) = \left(\frac{-5}{6}\right)(4!) = \boxed{-20}$$

[2]

Question 4: It can be shown that the Maclaurin series for $\sin^4(x)$ is

$$\sin^4(x) = x^4 - \frac{2x^6}{3} + \frac{x^8}{5} - \cdots$$

Use this to find the first three nonzero terms of the Maclauring series for $g(x) = 4\sin^3(x)\cos(x)$.

$$g(x) = \frac{d}{dx}\left[\sin^4(x)\right] = \frac{d}{dx}\left[x^4 - \frac{2x^6}{3} + \frac{x^8}{5} - \dots\right]$$

$$= 4x^3 - \frac{12x^5}{3} + \frac{8x^7}{5} - \dots$$

$$= \boxed{4x^3 - 4x^5 + \frac{8}{5}x^7 - \dots}$$

[4]

Question 5: Evaluate the following limit:

$$\lim_{x\to 0} \frac{\ln(1+x^2) - e^{(x^2)} + 1}{2x^4} \sim \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\left[\cancel{x^2} - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \cdots\right] - \left[\cancel{1} + \cancel{x^2} + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3!} + \cdots\right] \cancel{+1}}{2x^4}$$

$$= \lim_{x\to 0} \frac{-x^4 + \frac{x^6}{6} + \cdots}{2x^4}$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left[-1 + \frac{x^2}{6} + \cdots\right]}{2\cancel{x^4}}$$

$$= \boxed{-\frac{1}{2}}$$

[5]

Question 6: Find the the first four nonzero terms of the Taylor series for $h(x) = \dfrac{1}{2-3x}$ about $a = 1$ and state the open interval of convergence.

$$h(x) = \frac{1}{2-3x} = \frac{1}{2-3(x-1)-3} = \frac{1}{-1-3(x-1)}$$

$$= \frac{-1}{1-[-3(x-1)]}$$

$$= -\left[1 + (-3(x-1)) + (-3(x-1))^2 + \cdots\right]$$

$$= \boxed{-1 + 3(x-1) - 3^2(x-1)^2 + 3^3(x-1)^3 - \cdots}$$

Converges for $|-3(x-1)| < 1$

$\Rightarrow |x-1| < \frac{1}{3}$

$\Rightarrow \frac{2}{3} < x < \frac{4}{3}$

$$\boxed{\therefore I = \left(\frac{2}{3}, \frac{4}{3}\right)}$$

[5]

Question 7: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{4^k (x-3)^{2k}}{k^2}$$

$$u_k(x) = \frac{4^k (x-3)^{2k}}{k^2}$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{4^{k+1}}{4^k} \cdot \frac{k^2}{(k+1)^2} \cdot \frac{(x-3)^{2(k+1)}}{(x-3)^{2k}} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| 4 \cdot \underbrace{\left(\frac{k}{k+1}\right)^2}_{\to 1} \cdot (x-3)^2 \right| < 1$$

$$\Rightarrow 4|x-3|^2 < 1$$

$$\Rightarrow |x-3| < \frac{1}{2}$$

$$\boxed{\therefore R = \frac{1}{2}, \quad \mathcal{I} = \left(\frac{5}{2}, \frac{7}{2}\right)}$$

[5]

Question 8: Find the radius of convergence R and open interval of convergence $\mathcal{I}$ for the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x+1)^k}{2^k k!}$$

$$u_k(x) = \frac{(-1)^k (x+1)^k}{2^k k!}$$

$$\lim_{k\to\infty} \left| \frac{u_{k+1}(x)}{u_k(x)} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(-1)^{k+1}}{(-1)^k} \cdot \frac{2^k}{2^{k+1}} \cdot \frac{k!}{(k+1)!} \cdot \frac{(x+1)^{k+1}}{(x+1)^k} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \left| (-1)\left(\frac{1}{2}\right)\left(\frac{1}{k+1}\right)(x+1) \right| < 1$$

$\Rightarrow \quad 0 < 1$: True for every x, so

$$\boxed{R = \infty, \ \mathcal{I} = (-\infty, \infty)}$$

[5]