Question 1: Let

$$g(x) = \begin{cases} x^4 - cx^2 & \text{if } x < 2 \\ c^2x + 18 & \text{if } x \geq 2 \end{cases}$$

Find the constant c that makes g continuous at all real numbers.

For $c \neq 2$ g is defined by polynomial functions, so is continuous.

At $c = 2$ we need

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2)$$

$$\Rightarrow \lim_{x \to 2^-} (x^4 - cx^2) = \lim_{x \to 2^+} (c^2x + 18) = c^2(2) + 18$$

$$\Rightarrow 2^4 - c(2^2) = c^2(2) + 18 = c^2(2) + 18$$

$\therefore\ 16 - 4c = 2c^2 + 18$

$\therefore\ 2c^2 + 4c + 2 = 0$

$\therefore\ 2(c^2 + 2c + 1) = 0$

$\therefore\ 2(c+1)^2 = 0$

$\therefore\ \boxed{c = -1}$

[5]

Question 2: Use the Intermediate Value Theorem to show that the equation $\sqrt{\frac{x}{\pi}} = \cos\left(\frac{x}{2}\right)$ has a solution on the interval $[0, \pi]$.

Let $f(x) = \sqrt{\frac{x}{\pi}} - \cos\left(\frac{x}{2}\right)$.

Must show that $f(c) = 0$ for some $0 \leq c \leq \pi$.

$f(0) = \sqrt{\frac{0}{\pi}} - \cos\left(\frac{0}{2}\right) = -1$.

$f(\pi) = \sqrt{\frac{\pi}{\pi}} - \cos\left(\frac{\pi}{2}\right) = 1$.

Since f is continuous on $[0, \pi]$ and $f(0) < 0 < f(\pi)$, there is some $0 < c < \pi$ such that $f(c) = 0$ by the Intermediate Value Theorem.

[5]

Question 3: Evaluate the following limits, if they exist. If a limit does not exist because it is $+\infty$ or $-\infty$, state which with an explanation of your reasoning. (Do not use L'Hospital's rule to evaluate limits.)

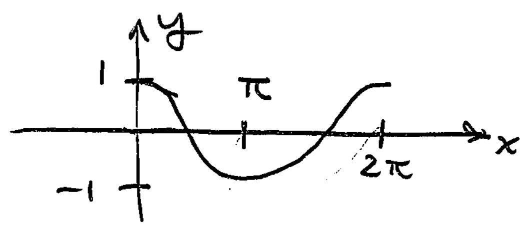
(a) $\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}}$ $\sim$ "$\frac{\infty}{\infty}$"

$$= \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4\left(1+\frac{1}{x^4}\right)}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}}{\cancel{x^2}\sqrt{1+\frac{1}{x^4}}} = \frac{1}{1} = \boxed{1}$$

[3]

(b) $\lim_{x\to\pi} \frac{\cancel{\sin x}}{\cancel{(x-\pi)^2}} \frac{\cos(x)}{(x-\pi)^3}$ $y = \cos x$:



As $x \to \pi^-$, $\cos(x) \to -1$ while $(x-\pi)^3 \to 0^-$,

$\therefore \lim_{x\to\pi^-} \frac{\cos(x)}{(x-\pi)^3} = +\infty$.

As $x \to \pi^+$, $\cos(x) \to -1$ while $(x-\pi)^3 \to 0^+$

$\therefore \lim_{x\to\pi^+} \frac{\cos(x)}{(x-\pi)^3} = -\infty$

$\therefore \lim_{x\to\pi} \frac{\cos(x)}{(x-\pi)^3}$ Does not exist!

[3]

(c) $\lim_{x\to\infty} \sqrt{9x^2+1} - 3x$

$\sim \infty - \infty$

$$= \lim_{x\to\infty} \frac{\sqrt{9x^2+1} - 3x}{1}\left(\frac{\sqrt{9x^2+1}+3x}{\sqrt{9x^2+1}+3x}\right)$$

$$= \lim_{x\to\infty} \frac{\cancel{9x^2}+1-\cancel{9x^2}}{\sqrt{9x^2+1}+3x\,\}\to\infty}$$

$$= \boxed{0}$$

[4]

Question 4:

(a) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{1}{1+x^2}$. Neatly show all steps and use proper notation. (No credit will be given if $f'(x)$ is found using derivative rules.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{1+(x+h)^2} - \frac{1}{1+x^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(1+x^2) - (1+(x+h)^2)}{(1+(x+h)^2)(1+x^2)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\cancel{1}+\cancel{x^2}-\cancel{1}-\cancel{x^2}-2xh-h^2}{(1+(x+h)^2)(1+x^2)}$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}(-2x-h)}{(1+(x+h)^2)(1+x^2)}\right]$$

$$= \boxed{\frac{-2x}{(1+x^2)^2}}$$

[8]

(b) Now check your work in part (a) by finding $\dfrac{d}{dx}\left[\dfrac{1}{1+x^2}\right]$ using derivative rules.

$$\frac{d}{dx}\left[\frac{1}{1+x^2}\right] = \frac{d}{dx}\left[(1+x^2)^{-1}\right] = -(1+x^2)^{-2}(2x)$$

chain!

$$= \boxed{\frac{-2x}{(1+x^2)^2}}$$

[2]

Question 5: A ball with an initial velocity of 5 m/s rolls down a hill. The position of the ball after t seconds is $s(t) = 5t + 3t^2$ metres. How long does it take the velocity to reach 35 m/s?

Solve $v(t) = 35$ for t.

$v(t) = s'(t) = 5 + 6t$

$5 + 6t = 35$

$\Rightarrow 6t = 30$

$\Rightarrow \boxed{t = 5 \text{ seconds}}$

[3]

Question 6: Determine $q''(0)$ if $q(t) = \sec(t)$

$q'(t) = \sec(t) \cdot \tan(t)$

$q''(t) = \sec(t)\tan(t)\tan(t) + \sec(t) \cdot \sec^2(t)$

$q''(0) = \underbrace{\sec(0)}_{1}\underbrace{\tan(0)}_{0}\underbrace{\tan(0)}_{0} + \underbrace{\sec(0)}_{1}\underbrace{\sec^2(0)}_{1}$

$= \boxed{1}$

[3]

Question 7: Find an equation of the tangent line to $y = \sqrt{1 + 4\sin(x)}$ at the point where $x = 0$.

(i) Point on line: At $x = 0$, $y = \sqrt{1 + 4\sin(0)} = 1$

$\therefore (0, 1)$.

(ii) slope of line: $m = \left.\frac{dy}{dx}\right|_{x=0} = \frac{d}{dx}\left[(1 + 4\sin(x))^{\frac{1}{2}}\right]_{x=0}$

$= \frac{1}{2}(1 + 4\sin(x))^{-\frac{1}{2}} \cdot 4\cos(x)\Big|_{x=0}$

$= \boxed{2}$

$\therefore$ Equation is

$y - 1 = 2(x - 0)$ or $\boxed{y = 2x + 1}$

[4]

Question 8: Find the following derivatives (it is not necessary to simplify your answers):

(a) $y = \frac{1+\sin(x)}{x^2}$

$$\boxed{y' = \frac{x^2\cos(x) - (1+\sin(x))(2x)}{(x^2)^2}}$$

[2]

(b) $f(x) = \left(\sqrt{x} + \frac{3}{x}\right)\tan(x)$ $= \left(x^{1/2} + \frac{3}{x}\right)\tan(x)$

$$\boxed{f'(x) = \left(\tfrac{1}{2}x^{-\frac{1}{2}} - \frac{3}{x^2}\right)\tan(x) + \left(x^{1/2} + \frac{3}{x}\right)\sec^2(x)}$$

[2]

(c) $y = \frac{x}{\sqrt{7-3x}}$ $= x(7-3x)^{-\frac{1}{2}}$

$$y' = (1)(7-3x)^{-\frac{1}{2}} + x\left(-\tfrac{1}{2}\right)(7-3x)^{-3/2}(-3)$$

$$= \boxed{(7-3x)^{-\frac{1}{2}} + \left(\frac{3x}{2}\right)(7-3x)^{-3/2}}$$

[3]

(d) $g(t) = \sin(\cos(\tan(t^3)))$

$$\boxed{g'(t) = \cos(\cos(\tan(t^3))) \cdot (-\sin(\tan(t^3))) \cdot \sec^2(t^3) \cdot 3t^2}$$

[3]