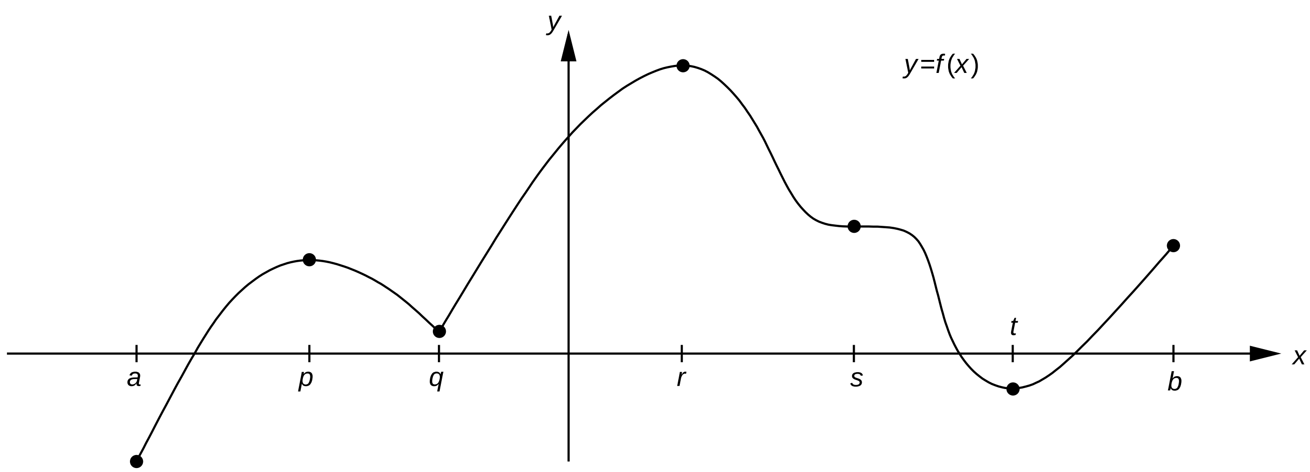
First Derivatives and Shapes of Curves

We continue our work on using derivatives to study graphs of functions. So far we have seen how derivatives can be used to identify the absolute extrema of a continuous function on a closed interval. In this lesson we ask: What information does $f'(x)$ give us about the shape of the graph of $y = f(x)$? Begin with some terminology. Consider the graph of a general function f with domain $D = [a, b]$:



Recall:

- A function f is said to be **increasing** on an interval if for any numbers $x_1 < x_2$ from the interval, $f(x_1) < f(x_2)$. If a function is increasing on an interval then its graph rises as x increases.

  In the graph above f is increasing on $[a, p]$, $[q, r]$ and $[t, b]$.

- A function f is said to be **decreasing** on an interval if for any numbers $x_1 < x_2$ from the interval, $f(x_1) > f(x_2)$. If a function is decreasing on an interval then its graph falls as x increases.

  In the graph above f is decreasing on $[p, q]$ and $[r, t]$.

Using derivatives we can easily determine the intervals of increase and decrease of a function.

Increasing/Decreasing Test

Recall

$$f'(c) = \text{slope of the tangent line to graph of } y = f(x) \text{ at } x = c\ ,$$

so

$$\begin{aligned} f'(c) > 0 &\Rightarrow \text{outputs of } f \text{ are increasing as } x \text{ passes through } c \\ &\Rightarrow \text{graph of } f \text{ is rising as } x \text{ passes through } c \\ f'(c) < 0 &\Rightarrow \text{outputs of } f \text{ are decreasing as } x \text{ passes through } c \\ &\Rightarrow \text{graph of } f \text{ is falling as } x \text{ passes through } c \end{aligned}$$

This gives the **Test for Intervals of Increase and Decrease of a Function:**

(i) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(ii) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Observe on our graph: whenever f changes from increasing to decreasing, or vice versa, it does so at a critical number ($x = p, q, r$ and t). However, not every critical number corresponds to such a change: f is decreasing on both sides of $x = s$. Putting all of this together:

To determine the intervals of increase and decrease of a function f:

(i) Find points at which f' changes sign (from positive to negative or vice versa). f' can change sign at

- critical numbers: x-values at which $f'(x) = 0$ or $f'(x)$ does not exist
- values of x at which f itself is not defined

(ii) Test $f'(x)$ on the subintervals defined by the points from (i).

Once you have determined the intervals of increase/decrease of f, it is easy to read off the relative extrema (that is, the relative maxima and minima) using

The First Derivative Test: Suppose $x = c$ is a critical number of a continuous function f.

(i) If f' changes from positive to negative at $x = c$, then f has a relative maximum of $f(c)$ at $x = c$.

(ii) If f' changes from negative to positive at $x = c$, then f has a relative minimum of $f(c)$ at $x = c$.

(iii) If f' does not change sign at $x = c$, then f has a neither a relative maximum nor relative minimum at $x = c$.