Question 7: Find the following derivatives (it is not necessary to simplify your answers):

(a) $y = \sqrt{1 + 2e^{3x}}$ $= (1+2e^{3x})^{\frac{1}{2}}$

$$y' = \frac{1}{2}(1+2e^{3x})^{-\frac{1}{2}}(6e^{3x}) \leftarrow \text{OK too!}$$

$$= \boxed{\frac{3e^{3x}}{\sqrt{1+2e^{3x}}}}$$

[2]

(b) $f(x) = 10^{1-x^2}$

$$f'(x) = 10^{1-x^2}\ln(10)(-2x) \leftarrow \text{OK too!}$$

$$= \boxed{-2\ln(10)\,x \cdot 10^{1-x^2}}$$

[2]

(c) $y = x^2 \ln(2x + 1)$

$$y' = 2x\ln(2x+1) + x^2\frac{1}{2x+1} \cdot 2 \leftarrow \text{OK too!}$$

$$= \boxed{\frac{2x(2x+1)\ln(2x+1) + 2x^2}{2x+1}}$$

[3]

(d) $g(t) = [\ln(1 + e^{3t})]^2$

$$g'(t) = 2\ln(1+e^{3t}) \cdot \frac{1}{1+e^{3t}} \cdot e^{3t} \cdot 3 \leftarrow \text{OK too!}$$

$$= \boxed{\frac{6e^{3t}\ln(1+e^{3t})}{1+e^{3t}}}$$

[3]

Question 1: Find an equation of the tangent line to the curve

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point $(0, 1/2)$.

$$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}\left[(2x^2+2y^2-x)^2\right]$$

$$2x + 2yy' = 2(2x^2+2y^2-x)(4x+4yy'-1)$$

at $(0, \frac{1}{2})$:

$$(2)(0)+(2)(\tfrac{1}{2})y' = 2(2(0^2)+2(\tfrac{1}{2})^2-0)((4)(0)+(4)(\tfrac{1}{2})y'-1)$$

$$y' = 2(\tfrac{1}{2})(2y'-1)$$

$$y' = 2y'-1$$

$$\therefore y' = 1$$

$\therefore$ Equation of tangent line is

$$\boxed{y-\tfrac{1}{2} = 1\cdot(x-0)} \text{ or } \boxed{y = x+\tfrac{1}{2}}$$

[5]

Question 2: Find y'' by implicit differentiation if

$$x^3 + y^3 = 1$$

Express your answer as a single simplified fraction involving the variables x and y only.

$$\frac{d}{dx}[x^3+y^3] = \frac{d}{dx}[1]$$

$$3x^2 + 3y^2y' = 0$$

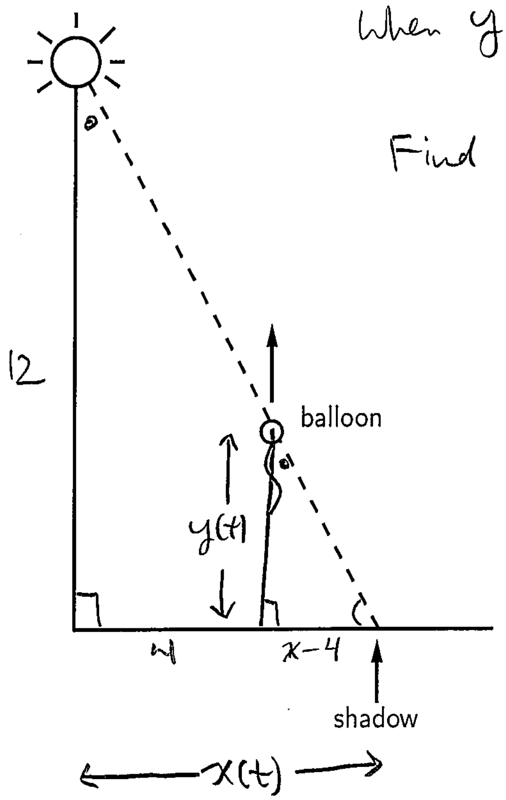
$$\therefore y' = -\frac{3x^2}{3y^2} = \frac{-x^2}{y^2}$$

$$\therefore y'' = \frac{d}{dx}[y'] = \frac{d}{dx}\left[-\frac{x^2}{y^2}\right] = \frac{-2xy^2+x^2 2yy'}{y^4}$$

$$\therefore y'' = \frac{-2xy^2+2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4} = \frac{-2x\cancel{y^4}y^3-2x^4\cancel{y}}{\cancel{y^6}y^5} = \boxed{\frac{-2x(x^3+y^3)}{y^5}}$$

[5]

Question 3: A balloon is released from ground level 4 m from the base of a 12 m tall lamppost. As the balloon rises vertically it casts a shadow on the ground as a result of the light atop the lamppost. When the balloon is 3 m above the ground it is rising at 1 m/s. At what rate is the shadow moving along the ground at that same instant?



When $y = 3$ m $\frac{dy}{dt} = +1 \ \frac{m}{s}$.

Find $\frac{dx}{dt}$ when $y = 3$.

By similar triangles,

$$\frac{x}{12} = \frac{x-4}{y}$$

$$\therefore \ xy = 12x - 48$$

$$xy - 12x = -48$$

$$x(y-12) = -48$$

$$x = \frac{-48}{y-12}$$

$$\therefore \ \frac{dx}{dt} = \frac{+48}{(y-12)^2}\frac{dy}{dt}.$$

When $y = 3$: $\frac{dx}{dt} = \frac{+48}{(3-12)^2} \cdot 1$

$$= \frac{+\cancel{48}\,16}{\cancel{81}\,27}$$

$$= +\frac{16}{27} \ \frac{m}{s}$$

$\boxed{\therefore \text{ Shadow is moving along ground at } \frac{16}{27} \ \frac{m}{s}.}$ **[10]**

Question 4: Use a linear approximation to approximate $\sqrt{26}$.

$f(x) = \sqrt{x}$, $a = 25$, $f(a) = \sqrt{25} = 5$

$f'(x) = \frac{1}{2\sqrt{x}}$; $f'(a) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$

$\therefore L(x) = f(a) + f'(a)(x-a) = 5 + \frac{1}{10}(x-25)$.

$\therefore \sqrt{26} = f(26) \approx L(26)$

$= 5 + \frac{1}{10}(26-25)$

$= \boxed{\frac{51}{10} \text{ or } 5.1}$

[5]

Question 5: The circumference of a sphere was measured to be 100 cm with a possible measurement error of 1/2 cm. Estimate the maximum error in the calculated surface area.
(Note: the surface area of a sphere of radius r is $S = 4\pi r^2$.)

$C = 2\pi r$

$\therefore dC = 2\pi\, dr$

$dC = \frac{1}{2}$, 50

$dr = \frac{1}{2\pi} dC = \frac{1}{4\pi}$

$S = 4\pi r^2$

$\therefore dS = 4\pi \cdot 2r\, dr$

$\left\{ \begin{array}{l} C = 100, 50 \\ r = \frac{50}{\pi} \text{ cm.} \end{array} \right.$

$= \cancel{4\pi} \cdot 2 \cdot \frac{50}{\pi} \cdot \frac{1}{\cancel{4\pi}}$

$= \boxed{\frac{100}{\pi} \text{ cm}^2}$

[5]

Question 6:

(a) Find the domain of $f(x) = \dfrac{x}{1 - e^{x-2}}$

Here $1 - e^{x-2} \neq 0$, so $x - 2 \neq 0$, so $x \neq 2$.

$\therefore$ Domain is $(-\infty, 2) \cup (2, \infty)$.

[3]

(b) Find the limit: $\lim\limits_{x \to 3^-} e^{5/(3-x)}$

As $x \to 3^-$, $3 - x \to 0^+$, so $\frac{5}{3-x} \to +\infty$

$\therefore \lim\limits_{x \to 3^-} e^{\frac{5}{3-x}} = \boxed{+\infty}$.

[3]

(c) Express as a single simplified logarithm:

$$\left(\frac{1}{3}\right)\ln(x+2)^3 + \frac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right]$$

$$= \ln(x+2)^{3 \cdot \frac{1}{3}} + \left(\tfrac{1}{2}\right)\ln(x) - \left(\tfrac{1}{2}\right)\ln(x^2+3x+2)^2$$

$$= \ln(x+2) + \ln(x^{1/2}) - \ln(x^2+3x+2)^{2 \cdot \frac{1}{2}}$$

$$= \ln(x+2) + \ln\sqrt{x} - \ln(x^2+3x+2)$$

$$= \ln\left[\frac{(x+2)\sqrt{x}}{(x^2+3x+2)}\right] \longleftarrow \text{OK too!}$$

$$= \ln\left[\frac{\cancel{(x+2)}\sqrt{x}}{\cancel{(x+2)}(x+1)}\right] = \boxed{\ln\left[\frac{\sqrt{x}}{x+1}\right]}$$

[4]