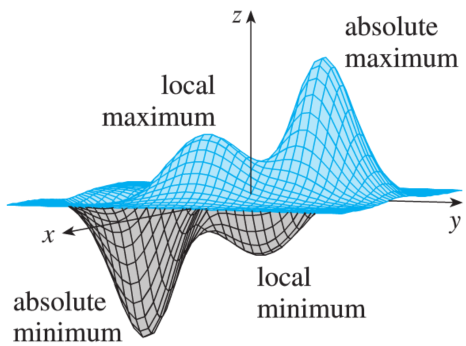
11.7: Maximum and Minimum Values:

Goal: Use partial derivatives to locate maximum and minimum values of functions of two variables:



Definition:

- f has a **local maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for every (x, y) in some disk with centre (a, b).
- f has a **local minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for every (x, y) in some disk with centre (a, b).
- f has an **absolute maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for every (x, y) in the domain of f.
- f has an **absolute minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for every (x, y) in the domain of f.

Note: the term *relative minimum* (resp. *maximum*) is equivalent to *local minimum* (resp. *maximum*). The term *global minimum* (resp. *maximum*) is equivalent to *absolute minimum* (resp. *maximum*).

Theorem: If f has a local maximum or minimum at (a, b) and both $f_x(a, b)$, $f_y(a, b)$ exist, then $f_x(a, b) = f_y(a, b) = 0$.

Proof: (in the case of local maximum) If f has a local maximum at (a, b) then $f(x, b)$ has a local maximum at $x = a$ as a function of one variable, so either $f_x(a, b) = 0$ or $f_x(a, b)$ does not exist. Since $f_x(a, b)$ exists by hypothesis it must be that $f_x(a, b) = 0$. By a similar argument $f_y(a, b) = 0$.

Definition: A point (a, b) is a **critical point** of f if $f_x(a, b) = f_y(a, b) = 0$ of if at least one of $f_x(a, b)$, $f_y(a, b)$ fails to exist.

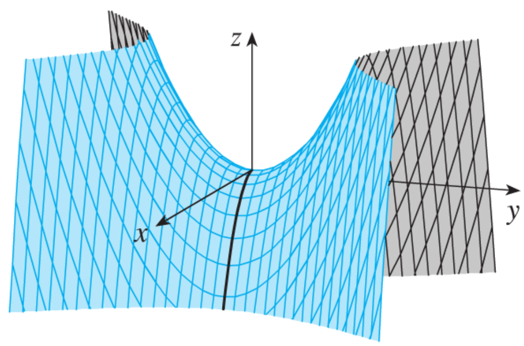
Conclusion: Local extrema occur at critical points, but not every critical point corresponds to a local extremum.

As in single variable calculus, the nature of critical points can be determined in part using

Theorem (Second Derivative Test): Suppose that f_{xx}, f_{yy}, f_{xy} and f_{yx} are all continuous on a disk with centre (a, b), and that $f_x(a, b) = f_y(a, b) = 0$. Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
- If $D < 0$ then $f(a, b)$ is neither a local minimum nor maximum; (a, b) is a *saddle point* (the graph of f crosses its tangent plane at (a, b)):



- If $D = 0$ then the test fails.

  Note: D above is sometimes called the **Hessian** and the matrix $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ the *Hessian matrix*.

Example: Determine the local extreme values of $f(x, y) = (2x - x^2)(2y - y^2)$.