Question 1: Find the volume of the largest rectangular box in the first octant having three faces on the coordinate planes and one vertex on the plane $x+2y+3z=6$. You may use any method you like, but be sure to justify that your solution is indeed the maximum.

(1) By Method of Lagrange Multipliers:

Maximize $f(x,y,z)=xyz$

subject to $g(x,y,z)=x+2y+3z=6$.

$\nabla f=\lambda\nabla g \Rightarrow$ ① $yz=\lambda$ ② $xz=2\lambda$ ③ $xy=3\lambda$ ④ $x+2y+3z=6$

At the maximum $xyz\neq 0$, so $x\neq 0, y\neq 0, z\neq 0 \Rightarrow \lambda\neq 0$.

② ÷ ① $\Rightarrow \frac{x}{y}=2 \Rightarrow x=2y$.

③ ÷ ② $\Rightarrow \frac{y}{z}=\frac{3}{2} \Rightarrow z=\frac{2}{3}y$.

$\therefore$ ④ $\Rightarrow 2y+2y+3(\frac{2}{3}y)=6 \Rightarrow y=1, \therefore x=2, z=\frac{2}{3}$.

$\therefore$ Maximum of $f(x,y,z)=xyz=(2)(1)(\frac{2}{3})=\boxed{\frac{4}{3}}$

Justification: f has an abs. max. on the plane within the first octant, and this abs. max. also corresponds to a relative max. at which the Lagrange multiplier equations must be satisfied. We found a single such point, so this point must correspond to the abs. max.

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(2) By Conventional constrained optimization:

Maximize $f(x,y,z)=xyz$ ①

Subject to $g(x,y,z)=x+2y+3z=6$ ②

② $\Rightarrow x=6-2y-3z$

① $\Rightarrow f(x,y,z)=h(y,z)=(6-2y-3z)yz$

$=6yz-2y^2z-3yz^2$

$\frac{\partial h}{\partial y}=6z-4yz-3z^2=z(6-4y-3z)=0$ ③

$\frac{\partial h}{\partial z}=6y-2y^2-6yz=y(6-2y-6z)=0$ ④

Noting that $xyz\neq 0$,

③ $\Rightarrow z=\frac{6-4y}{3}$,

$\therefore$ ④ $\Rightarrow 6-2y-\cancel{6}^{2}\left(\frac{6-4y}{\cancel{3}}\right)=0$

$\Rightarrow y=1 \Rightarrow z=\frac{2}{3} \Rightarrow x=2$

$\therefore f(2,1,\frac{2}{3})=\boxed{\frac{4}{3}}$ is the abs. max.

Justification: f has an abs. max. within the first octant which is also a relative max and, hence corresponds to a critical point. We found a single such point, so this point must correspond to the abs. max. [10]

Question 2: Compute $\iint_R xye^{xy^2}\,dA$ where $R = [0,2] \times [0,1]$.

$$\iint_R xye^{xy^2} = \frac{1}{2}\int_0^2 \int_0^1 2xy\,e^{xy^2}\,dy\,dx$$

$$= \frac{1}{2}\int_0^2 \left[e^{xy^2}\right]_0^1 dx$$

$$= \frac{1}{2}\int_0^2 (e^x - 1)\,dx$$

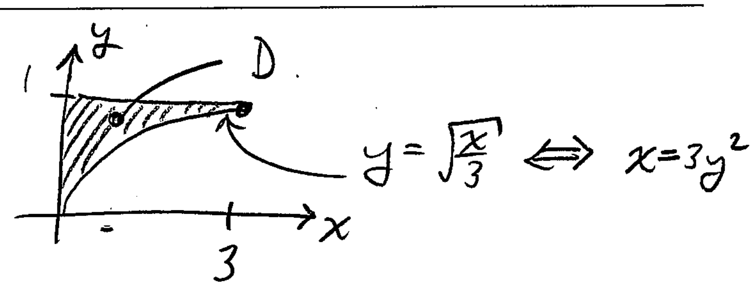
$$= \frac{1}{2}\left[e^x - x\right]_0^2$$

$$= \frac{e^2 - 2 - 1}{2}$$

$$= \boxed{\frac{e^2-3}{2}}$$

[5]

Question 3: Compute $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx$



$$\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\,dy\,dx$$

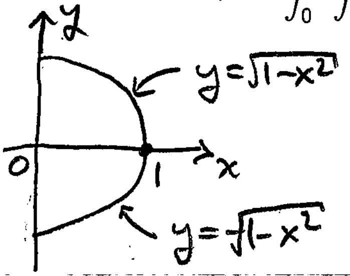
$$= \int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy$$

$$= \int_0^1 3y^2 e^{y^3}\,dy$$

$$= \left[e^{y^3}\right]_0^1 = \boxed{e-1}$$

[5]

Question 4: Compute $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \cos(x^2+y^2+1)\,dydx$ (polar coordinates may help here).



$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \cos(x^2+y^2+1)\,dy\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \cos(1+r^2)\, r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \left[\frac{\sin(1+r^2)}{2}\right]_0^1 d\theta$$

$$= \boxed{\left(\frac{\sin(2)-\sin(1)}{2}\right)\pi}$$

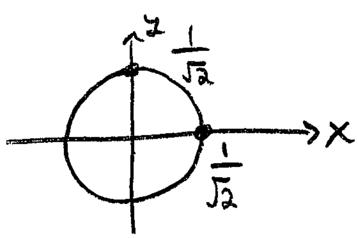
[5]

Question 5: Find the volume of the region that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.

For domain D, project curve of intersection of cone & sphere onto xy-plane: $x^2+y^2+z^2=1 \Rightarrow x^2+y^2+(\sqrt{x^2+y^2})^2=1$

$$\Rightarrow x^2+y^2=\frac{1}{2}=\left(\frac{1}{\sqrt{2}}\right)^2$$

$\therefore$ D is



$$\therefore V=\iint_D \sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\,dA$$

$$= \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left[(1-r^2)^{1/2}-r\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{(1-r^2)^{3/2}}{3}\Bigg|_0^{1/\sqrt{2}} - \frac{r^3}{3}\Bigg|_0^{1/\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}\left(1-\left(\frac{1}{2}\right)^{3/2}-\left(\frac{1}{2}\right)^{3/2}\right) d\theta$$

$$= \left(\frac{\sqrt{2}-1}{3\sqrt{2}}\right)(2\pi) = \boxed{\left(\frac{2-\sqrt{2}}{3}\right)\pi}$$

[5]

Question 6: Compute $\iiint_E e^{z/y}\,dV$ where $E = \{(x,y,z) \mid 0 \le y \le 1,\ \ y \le x \le 1,\ \ 0 \le z \le xy\}$.

$$\iiint_E e^{z/y}\,dV = \int_{y=0}^{1}\int_{x=y}^{1}\int_{z=0}^{xy} e^{z/y}\,dz\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=y}^{1} \left[y e^{z/y} \right]_0^{xy} dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=y}^{1} y(e^{x}-1)\,dx\,dy$$

$$= \int_{y=0}^{1} y(e^{x}-x)\Big|_y^1\,dy$$

$$= \int_0^1 y(e-1-e^{y}+y)\,dy$$

$$= \int_0^1 (e-1)y + y^2 - ye^{y}\,dy$$

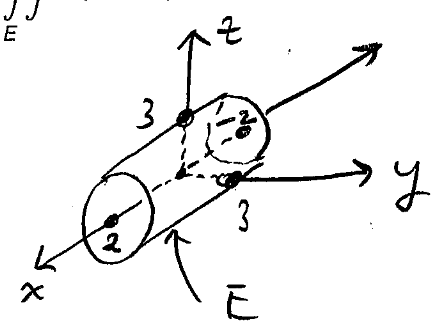
$$= \frac{(e-1)}{2}y^2\Big|_0^1 + \frac{y^3}{3}\Big|_0^1 - \left[ye^{y} - e^{y}\right]_0^1$$

$$= \left(\frac{e-1}{2}\right) + \frac{1}{3} - \cancel{e} + \cancel{e} - 1$$

$$= \boxed{\frac{e}{2} - \frac{7}{6}}$$

[5]

Question 7: Suppose E is the solid region bounded by the surfaces $y^2+z^2=9$, $x=-2$ and $x=2$. Express $\iiint_E f(x,y,z)\,dV$ as an iterated integral in the order $dz\,dy\,dx$.



$$y^2+z^2=9 \Rightarrow z = \pm\sqrt{9-y^2}$$

$$\therefore \iiint_E f(x,y,z)\,dV$$

$$= \boxed{\int_{x=-2}^{2}\int_{y=-3}^{3}\int_{z=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx}$$

[5]