Question 1: Find the linear approximation of $f(x,y) = x^2e^{xy^2}$ at the point $(2,0)$ and use it to approximate $f(1.95,-0.1)$.

$$f_x(2,0) = 2xe^{xy^2} + x^2e^{xy^2}y^2 \Big|_{(2,0)} = 4$$

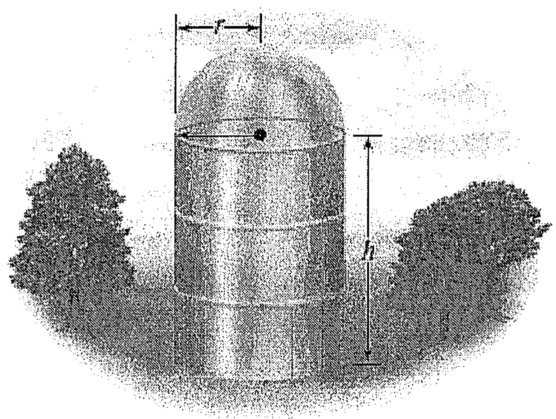
$$f_y(2,0) = x^2e^{xy^2}2xy \Big|_{(2,0)} = 0$$

$$\therefore L(x,y) = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0)$$
$$= 4 + 4(x-2) + 0(y-0)$$
$$= 4x - 4$$

$$\therefore f(1.95,-0.1) \approx L(1.95,-0.1) = 4(1.95) - 4 = \boxed{3.8}$$

[5]

Question 2: A steel grain silo is constructed by surmounting a hemisphere of radius r atop a cylinder of height h as shown below:



The floor of the silo is concrete. If the radius is 4 m and the height is 8 m, use differentials to estimate the total volume of steel in the silo walls and roof if the steel has a uniform thickness of 1/50 m .

$$V = \pi r^2 h + \left(\tfrac{1}{2}\right)\left(\tfrac{4}{3}\pi r^3\right) = \pi r^2 h + \tfrac{2}{3}\pi r^3$$

$$dV = V_h\,dh + V_r\,dr = \pi r^2 dh + (2\pi r h + 2\pi r^2)dr$$

Here $r=4$, $h=8$, $dr = \frac{1}{50}$, $dh = 0$ (since floor is concrete, not steel).

$$\therefore dV = \pi\cdot 4^2\cdot 0 + (2\cdot\pi\cdot 4\cdot 8 + 2\cdot\pi\cdot 4^2)\left(\tfrac{1}{50}\right) = \boxed{\tfrac{48\pi}{25}\ m^3}$$

If the floor were also made of steel $\frac{1}{50}$ m thick then $dh = \frac{1}{50}$, giving $dV = \frac{56\pi}{25}\ m^3$.

[5]

Question 3: Let

$$w = xy + yz^2, \quad x = e^t, \quad y = e^t \sin(t), \quad z = e^t \cos(t).$$

Find $\frac{dw}{dt}$ at $t = 0$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= y \cdot e^t + (x+z^2)\left[e^t \sin(t) + e^t \cos(t)\right] + 2yz\left[e^t\cos(t) - e^t \sin(t)\right]$$

At $t=0$, $x = e^0 = 1$, $y = e^0 \sin(0) = 0$, $z = e^0 \cos(0) = 1$

$$\therefore \left.\frac{dw}{dt}\right|_{t=0} = \cancel{0 \cdot e^0} + (1+1^2)\left[\cancel{e^0 \sin(0)} + \cancel{e^0\cos(0)}\right] + \cancel{(2)(0)(1)\left[e^0\cos(0) - e^0\sin(0)\right]}$$

$$= \boxed{2}$$

[3]

Question 4: Let

$$z = e^r \sin\theta, \quad r = s^3 t^2, \quad \theta = \sqrt{s^2 + t^2}$$

Find $\frac{\partial z}{\partial s}$ at $(s, t) = (0, \pi)$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= e^r \sin\theta \, 3s^2 t^2 + e^r \cos\theta \left(\frac{s}{\sqrt{s^2+t^2}}\right)$$

$$\therefore \left.\frac{\partial z}{\partial s}\right|_{(0,\pi)} = 0$$

[4]

Question 5: The radius of a right circular cone is increasing at 1 m/s while the height is decreasing at 2 m/s. At what rate is the volume changing when the radius is 20 m and the height is 30 m?

$$\frac{dr}{dt} = 1, \quad \frac{dh}{dt} = -2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt}$$

$$= \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

At $r = 20$, $h = 30$:

$$\frac{dV}{dt} = \left(\frac{2}{3}\right)(\pi)(20)(30)(1) + \left(\frac{1}{3}\right)(\pi)(20^2)(-2)$$

$$= 400\pi - \frac{800}{3}\pi$$

$$= \boxed{\frac{400\pi}{3} \ m^3/s}$$

[3]

Question 6: Find the directional derivative of $f(x, y, z) = xyz - y^2z^2\ln(x)$ at the point $P(1, 2, 1)$ in the direction of $Q(3, 3, 3)$.

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle 2,1,2\rangle}{\sqrt{2^2+1^2+2^2}} = \langle \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\rangle$$

$$D_{\vec{u}} f(1,2,1) = \nabla f(1,2,1) \cdot \vec{u}$$

$$= \left\langle yz - \frac{y^2z^2}{x}, xz - 2yz^2\ln(x), xy - 2y^2z\ln(x)\right\rangle\Big|_{(1,2,1)} \cdot \vec{u}$$

$$= \langle -2, 1, 2\rangle \cdot \langle \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\rangle$$

$$= \boxed{\tfrac{1}{3}}$$

[3]

Question 7: Find the direction in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at the point $(1, 0)$ has the value 1. State your answer using unit vectors.

Let $\vec{u} = \langle a, b\rangle$ where $a^2+b^2=1$

$$\nabla f(1,0) = \langle 2x + y\cos(xy), x\cos(xy)\rangle\Big|_{(1,0)} = \langle 2, 1\rangle.$$

$$D_{\vec{u}} f(1,0) = 1 \Rightarrow \langle 2,1\rangle \cdot \langle a,b\rangle = 1$$

$$\Rightarrow 2a+b=1$$

$$\Rightarrow b = 1-2a$$

Since $a^2+b^2=1$, $a^2+(1-2a)^2=1 \Rightarrow 5a^2-4a=0 \Rightarrow a(5a-4)=0$
$\Rightarrow a=0,\ a=\frac{4}{5}$

- $a=0 \Rightarrow b=1-2a=1 \Rightarrow \vec{u}_1 = \langle 0,1\rangle$
- $a=\frac{4}{5} \Rightarrow b=1-2a=-\frac{3}{5} \Rightarrow \vec{u}_2 = \langle \frac{4}{5}, -\frac{3}{5}\rangle$

[4]

Question 8: Find the equation of the tangent plane to the surface $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$ at the point $(-2, 1, -3)$.

$$F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3$$

$$\nabla F(-2,1,-3) = \left\langle \frac{x}{2}, 2y, \frac{2z}{9}\right\rangle\Big|_{(-2,1,-3)} = \langle -1, 2, -\tfrac{2}{3}\rangle$$

$$\nabla F(-2,1,-3) \cdot \langle x+2, y-1, z-(-3)\rangle = 0$$

$$\Rightarrow \langle -1, 2, -\tfrac{2}{3}\rangle \cdot \langle x+2, y-1, z+3\rangle = 0$$

$$\Rightarrow -(x+2) + 2(y-1) - \tfrac{2}{3}(z+3) = 0 \quad \text{or} \quad \boxed{3x-6y+2z+18=0}$$

[3]

Question 9: Determine all critical points of $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$ and classify each as corresponding to a local maximum, a local minimum or a saddle point.

$f_x(x,y) = 6x^2 + y^2 + 10x$

$f_y(x,y) = 2xy + 2y = 2y(x+1)$

$f_{xx}(x,y) = 12x + 10$

$f_{yy}(x,y) = 2x + 2$

$f_{xy} = 2y$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (12x+10)(2x+2) - (2y)^2$

$f_y(x,y) = 0 \Rightarrow y = 0, \quad x = -1$

At $y=0$: $f_x(x,y) = 0 \Rightarrow 6x^2 + 0^2 + 10x = 0$

$\Rightarrow 2x(3x+5) = 0$

$\Rightarrow x = 0, \ x = \frac{-5}{3}$

$\therefore (0,0), \left(\frac{-5}{3}, 0\right)$ are C.P.s

At $x = -1$: $f_x(x,y) = 0 \Rightarrow 6(-1)^2 + y^2 + 10(-1) = 0$

$\Rightarrow y^2 = 4$

$\Rightarrow y = 2, -2$

$\therefore (-1,2), (-1,-2)$ are C.P.s

C.P.	$D = (12x+10)(2x+2) - (2y)^2$	$f_{xx} = 12x + 10$	Conclusion
$(0,0)$	$20 > 0$	$10 > 0$	loc. min.
$\left(\frac{-5}{3}, 0\right)$	$\frac{40}{3} > 0$	$-10 < 0$	loc. max
$(-1,2)$	$-16 < 0$	—	Saddle
$(-1,-2)$	$-16 < 0$	—	Saddle.

[10]

Question 10: Find the points on the surface $y^2 = 9 + xz$ which are nearest the origin.

Minimize $\ell = \sqrt{x^2+y^2+z^2}$

subject to $y^2 = 9 + xz$.

To minimize ℓ it is sufficient to minimize

$g(x,y,z) = \ell^2 = x^2+y^2+z^2$.

Since $y^2 = 9+xz$, $g(x,y,z) = x^2+9+xz+z^2$

$= f(x,z)$ say.

$f_x = 2x + z$

$f_z = x + 2z$

$f_x = 0 \Rightarrow z = -2x$

$f_z = 0 \Rightarrow x+2(-2x) = 0 \Rightarrow x = 0$ & so $z = -2x = 0$.

$\therefore (x,z) = (0,0)$ is the only C.P. of $f(x,z)$.

$f(x,z)$ clearly has an abs. min. which must occur at a C.P., and since we found a single C.P. of $f(x,y)$ it must correspond to the abs. min.

$\therefore$ $x=0$, $z=0$, $y^2 = 9+xz = 9 \Rightarrow y = \pm 3$.

$\therefore$ Points on $y^2 = 9+xz$ nearest $(0,0,0)$

are $(0,3,0)$, $(0,-3,0)$.

[10]