Question 1: Determine the point of intersection with the xy-plane of the line through $P(2,4,6)$ that is perpendicular to the plane $x - y + 3z = 7$.

Direction vector of line is $\vec{v} = \langle 1,-1,3 \rangle$

$\therefore$ Equation of line is $\vec{r}(t) = \langle 2,4,6 \rangle + t\langle 1,-1,3 \rangle = \langle 2+t, 4-t, 6+3t \rangle$

Line intersects the xy-plane when $z=0$: $6+3t=0$

$\Rightarrow t = -2$,

$\therefore$ Point of intersection is $\vec{r}(-2) = \boxed{\langle 0,6,0 \rangle}$

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Question 2: Determine the equation of the plane through the point $(1,-1,1)$ which contains the line having symmetric equations $x = 2y = 3z$.

Need two points on line : If $y=3 \Rightarrow z=2$ and $x=6$

If $y=6 \Rightarrow z=4$ and $x=12$

$\therefore$ $P_1(6,3,2)$ & $P_2(12,6,4)$ are on line (and plane).

$P_3(1,-1,1)$ is also on the plane.

$\therefore$ normal to plane is $\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 3 & 2 \\ -5 & -4 & -1 \end{vmatrix} = \langle 5,-4,-9 \rangle$

Using P_1 & $\vec{n}$, equation is

$(\langle x,y,z \rangle - \langle 1,-1,1 \rangle) \cdot (5,-4,-9) = 0$

$\Rightarrow$ $\boxed{\begin{array}{l} 5(x-1) - 4(y+1) - 9(z-1) = 0 \\ \text{or } 5x - 4y - 9z = 0 \end{array}}$

[5]

Question 3: Determine the point of intersection of $\mathbf{u} = \langle 1,1,0\rangle + t\langle 1,-1,2\rangle$ with $\mathbf{v} = \langle 2,0,2\rangle + s\langle -1,1,0\rangle$.

At point of intersection the components must be equal.

$\therefore$
$$\left.\begin{aligned} 1+t &= 2-s \quad ① \\ 1-t &= 0+s \quad ② \\ 0+2t &= 2+0 \quad ③ \end{aligned}\right\}$$
$\therefore$ ③ $\Rightarrow t=1$ } This is all we need!
② $\Rightarrow s=0$ } use to check!

$\therefore$ point of intersection is $\langle 1,1,0\rangle + 1\cdot\langle 1,-1,2\rangle = \boxed{\langle 2,0,2\rangle}$ (using $\vec{u}$). Check, using $\vec{v}$: $\langle 2,0,2\rangle + 0\langle -1,1,0\rangle = \langle 2,0,2\rangle$

[5]

Question 4: Find all points at which the space curve $\mathbf{r}(t) = t\mathbf{i} + (2t - t^2)\mathbf{k}$ intersects the surface $z = x^2 + y^2$.

Substitute component functions into $z = x^2+y^2$ and solve for t:

$$z = x^2+y^2$$
$$2t-t^2 = t^2+0^2$$
$$2t^2-2t = 0$$
$$2t(t-1) = 0$$
$$t=0,\ t=1.$$

$\therefore$ Points are $\vec{r}(0) = \langle 0,0,0\rangle$ & $\vec{r}(1) = \langle 1,0,1\rangle$

[5]

Question 5: Consider the space curve $\mathbf{r}(t) = e^{(t^2)}\mathbf{i} - \mathbf{j} + \ln(1+3t)\mathbf{k}$. Find a unit vector that is orthogonal to both $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ at the point where $t = 0$.

$$r'(t) = 2te^{t^2}\hat{\imath} + \frac{3}{1+3t}\hat{k} \; ; \; \vec{r}'(0) = \langle 0, 0, 3 \rangle$$

$$r''(t) = \left(2e^{t^2} + 4t^2e^{t^2}\right)\hat{\imath} - \frac{9}{(1+3t)^2}\hat{\jmath} \; ; \; \vec{r}''(0) = \langle 2, 0, -9 \rangle .$$

Let $\vec{n} = \vec{r}'(0) \times \vec{r}''(0) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 3 \\ 2 & 0 & -9 \end{vmatrix} = \langle 0, 6, 0 \rangle$

Required unit vector is then $\vec{u} = \frac{\vec{n}}{|\vec{n}|} = \boxed{\langle 0, 1, 0 \rangle}$

[5]

Question 6: The position function of a particle is $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed a minimum?

$$\vec{v}(t) = \vec{r}'(t) = \langle 2t, 5, 2t - 16 \rangle .$$

Speed is $f(t) = |\vec{v}(t)| = \sqrt{(2t)^2 + 5^2 + (2t-16)^2}$.

$f(t)$ is minimized when $g(t) = (f(t))^2$ is.

$$g(t) = 4t^2 + 25 + 4t^2 - 64t + 256$$
$$= 8t^2 - 64t + 281 .$$

$g'(t) = 16t - 64 = 0$ at $t = 4$.

$g''(4) = 16 > 0$, so $t = 4$ corresponds to a local (& absolute) minimum.

$\therefore$ Speed is minimized when $\boxed{t = 4}$.

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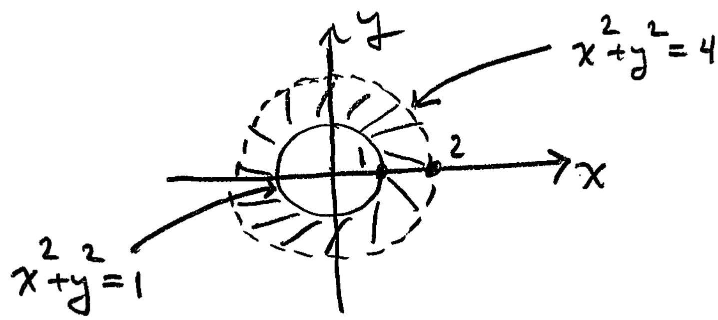
Question 7: Find and sketch the domain of $f(x, y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)$.

We must have $x^2+y^2-1 \geq 0$ & $4-x^2-y^2 > 0$

$\Rightarrow x^2+y^2 \geq 1$ $\Rightarrow x^2+y^2 < 4$

$\therefore$ domain is $\{(x,y) \mid 1 \leq x^2+y^2 < 4\}$



[5]

Question 8: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4 + y^2}$ does not exist. Carefully explain your reasoning.

- Letting $(x,y) \to (0,0)$ along x-axis, so $y=0$:

$$\frac{2x^2y}{x^4+y^2} = \frac{0}{x^4+0^2} \to 0 \text{ as } x \to 0.$$

- Letting $(x,y) \to (0,0)$ along curve $y=x^2$:

$$\frac{2x^2y}{x^4+y^2} = \frac{2x^2x^2}{x^4+(x^2)^2} = \frac{2}{1+1} = 1 \to 1 \text{ as } x \to 0.$$

Since different approach paths of (x,y) to $(0,0)$ yield different limiting values of $\frac{2x^2y}{x^4+y^2}$, the limit does not exist.

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Question 9: Let $u(t,w) = te^{w/t}$. Find and simplify $u_w(1,1) + u_{tt}(1,1) - u_{tw}(1,1)$.

$$u_w = te^{w/t} \cdot \frac{1}{t} = e^{\frac{w}{t}} \,;\; u_w(1,1) = e$$

$$u_t = e^{w/t} + te^{w/t} \cdot \left(\frac{-w}{t^2}\right) = e^{w/t} - \frac{w}{t}e^{\frac{w}{t}}$$

$$u_{tt} = -\frac{w}{t^2}e^{\frac{w}{t}} + \frac{w}{t^2}e^{\frac{w}{t}} + \frac{w^2}{t^3}e^{w/t} \,;\; u_{tt}(1,1) = -e + e + e = e$$

$$u_{tw} = u_{wt} = \frac{\partial}{\partial t}\left[e^{w/t}\right] = -\frac{w}{t^2}e^{w/t}$$ (using Clairaut's Thm).

$$u_{tw}(1,1) = -e$$

$$\therefore u_w(1,1) + u_{tt}(1,1) - u_{tw}(1,1) = e + e - (-e) = \boxed{3e.}$$

[5]

Question 10: Consider the surface S in $\mathbb{R}^3$ defined by $xy + yz + zx = 3$. Use implicit differentiation to calculate $z_x(1,1,1)$ and $z_y(1,1,1)$, and then use your result to state the equation of the tangent plane to S at the point $(1,1,1)$.

$$\frac{\partial}{\partial x}[xy + yz + zx] = \frac{\partial}{\partial x}[3]$$

$$y + y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \cdot x + z \cdot 1 = 0$$

at $(1,1,1)$: $$1 + 1 \cdot \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \cdot 1 + 1 \cdot 1 = 0$$

$$\Rightarrow 2\frac{\partial z}{\partial x} = -2$$

$$\Rightarrow \frac{\partial z}{\partial x} = -1$$

Similarly, $\frac{\partial z}{\partial y} = -1$ since $xy + yz + zx = 3$ is symmetric in x & y

$$z = z_0 + \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}(x - x_0) + \left.\frac{\partial z}{\partial y}\right|_{(1,1,1)}(y - y_0)$$

$$z = 1 + (-1)(x-1) + (-1)(y-1)$$

$$\boxed{\therefore x + y + z = 3}$$

[5]