Question 1. [10]:

(a) Find an equation of a sphere if one of its diameters has endpoints $(2,1,4)$ and $(4,3,10)$.

Centre is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

$= \left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right)$

$= (3, 2, 7)$

Radius is $r = \sqrt{(4-3)^2 + (3-2)^2 + (10-7)^2} = \sqrt{11}$

$\therefore$ Equation is $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$

$$\boxed{(x-3)^2 + (y-2)^2 + (z-7)^2 = 11}$$

[5]

(b) A sphere with centre $(-1,2,1)$ passes through the point $(6,-2,3)$. The sun, high in the sky directly above the sphere, shines directly down and casts a shadow of the sphere onto the xy-plane. Determine the equation of the circular boundary of the shadow.

Radius is $r = \sqrt{(6-(-1))^2 + (-2-2)^2 + (3-1)^2}$

$= \sqrt{69}$.

Centre of circle is $(x,y) = (-1,2)$

$\therefore$ Equation of circle is $(x-x_0)^2 + (y-y_0)^2 = r^2$

$$\boxed{(x+1)^2 + (y-2)^2 = 69}$$

[5]

Question 2. [10]:
For this question use the vectors

$$\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = 2\mathbf{j} - \mathbf{k}, \quad c = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

(a) Compute $|\mathbf{a} - 3\mathbf{c}|$.

$$|\vec{a} - 3\vec{c}| = \left|(2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}) - 3(\hat{\imath} + \hat{\jmath} + \hat{k})\right|$$
$$= |-\hat{\imath} - 7\hat{\jmath} + \hat{k}|$$
$$= \sqrt{(-1)^2 + (-7)^2 + (1)^2}$$
$$= \boxed{\sqrt{51}.}$$

[3]

(b) Find a unit vector in the same direction as $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

$$\vec{u} = \frac{|\vec{a} + \vec{b} + \vec{c}|}{|\vec{a} + \vec{b} + \vec{c}|} = \frac{3\hat{\imath} - \hat{\jmath} + 4\hat{k}}{|3\hat{\imath} - \hat{\jmath} + 4\hat{k}|}$$
$$= \frac{3\hat{\imath} - \hat{\jmath} + 4\hat{k}}{\sqrt{3^2 + (-1)^2 + 4^2}}$$
$$= \boxed{\frac{3}{\sqrt{26}}\hat{\imath} - \frac{1}{\sqrt{26}}\hat{\jmath} + \frac{4}{\sqrt{26}}\hat{k}}$$

[3]

(c) Determine the angle between **a** and the positive z-axis (if giving a decimal answer, round your answer to one decimal place and state units.)

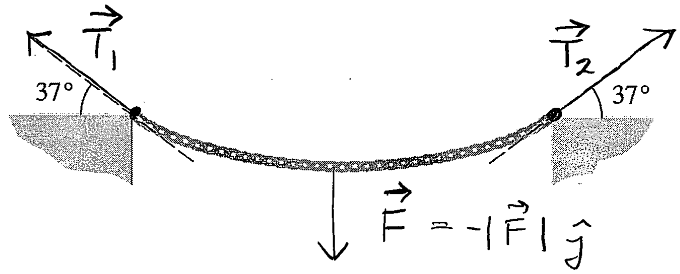
Equivalently, determine angle θ between $\vec{a} = \langle 2, -4, 4\rangle$ and $\hat{k} = \langle 0, 0, 1\rangle$.

$$\vec{a} \cdot \hat{k} = |\vec{a}|\,|\hat{k}| \cos(\theta)$$
$$\langle 2, -4, 4\rangle \cdot \langle 0, 0, 1\rangle = |\langle 2, -4, 4\rangle| \, \cancel{|\hat{k}|}^{1} \cos(\theta)$$
$$4 = \sqrt{2^2 + (-4)^2 + 4^2} \cos(\theta)$$
$$\therefore \cos(\theta) = \frac{4}{6} = \frac{2}{3}$$

[4]

$$\therefore \boxed{\theta = \arccos\left(\tfrac{2}{3}\right) \approx 0.8 \text{ radians or } 48.2^\circ}$$

Question 3. [10]: The tension forces $\mathbf{T}_1$ and $\mathbf{T}_2$ at the ends of the chain shown below each have magnitude 25 N. What is the weight of the chain? (Round your answer to one decimal place.)



$\vec{T}_1 = -|\vec{T}_1|\cos(37°)\hat{\imath} + |\vec{T}_1|\sin(37°)\hat{j}$

$\vec{T}_2 = |\vec{T}_2|\cos(37°)\hat{\imath} + |\vec{T}_2|\sin(37°)\hat{j}$

$|\vec{T}_1| = |\vec{T}_2| = 25\,N$ by symmetry,

and $\vec{T}_1 + \vec{T}_2 + \vec{F} = 0$

$\therefore\ |\vec{T}_1|\sin(37°)\hat{j} + |\vec{T}_2|\sin(37°)\hat{j} - |\vec{F}|\hat{j} = 0$

$\Rightarrow\ 2|\vec{T}_1|\sin(37°) = |\vec{F}|$

$\therefore\ |\vec{F}| = (2)(25)\sin(37) \approx \boxed{30.1\ N}$

[10]

Question 4. [10]:

(a) Determine the value of x so that the angle between $\langle 2,1,-1\rangle$ and $\langle 1,x,0\rangle$ is $\pi/4$.

$$\langle 2,1,-1\rangle \cdot \langle 1,x,0\rangle = |\langle 2,1,-1\rangle|\,|\langle 1,x,0\rangle| \cos\left(\tfrac{\pi}{4}\right)$$

$$\Rightarrow 2+x = \sqrt{6}\sqrt{1+x^2}\left(\tfrac{1}{\sqrt{2}}\right)$$

$$\Rightarrow 2+x = \sqrt{3}\sqrt{1+x^2}$$

$$\Rightarrow 4+4x+x^2 = 3+3x^2$$

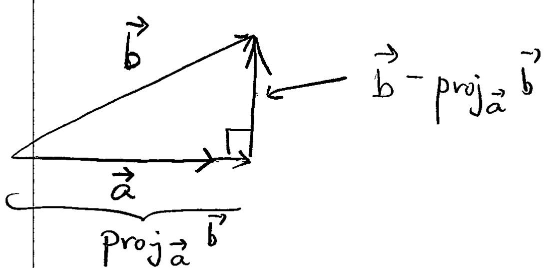
$$\Rightarrow 2x^2-4x-1=0$$

$$\Rightarrow x = \frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-1)}}{2(2)}$$

$$= \frac{4\pm 2\sqrt{6}}{4} = \boxed{\frac{2\pm\sqrt{6}}{2}}$$

[4]

(b) Simplify $\mathbf{a}\cdot(\mathbf{b}-\text{proj}_{\mathbf{a}}\mathbf{b})$ (Hint: draw a picture to see the answer).



$$\therefore\ \vec{a}\cdot(\vec{b}-\text{proj}_{\vec{a}}\vec{b}) = \boxed{0}$$

[4]

(c) Can $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$? If so, give an example, if not, explain why.

Yes! If $\vec{a}=\vec{b}$, then $\text{proj}_{\vec{a}}\vec{b} = \vec{b} = \vec{a} = \text{proj}_{\vec{b}}\vec{a}$

[2]

Question 5 [10]:

(a) Find a unit vector that is orthogonal to the plane containing the three points $P_1(0,0,-3)$, $P_2(4,2,0)$ and $P_3(3,3,1)$ (Note: these are points in $\mathbb{R}^3$, not vectors.)

$$\vec{u} = \frac{\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}}{|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}|}$$

$$\overrightarrow{P_1P_2} = \langle 4-0, 2-0, 0-(-3) \rangle = \langle 4, 2, 3 \rangle$$

$$\overrightarrow{P_1P_3} = \langle 3-0, 3-0, 1-(-3) \rangle = \langle 3, 3, 4 \rangle$$

$$\therefore \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix} = \langle -1, -7, 6 \rangle$$

$$|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}| = |\langle -1, -7, 6 \rangle| = \sqrt{(-1)^2 + (-7)^2 + 6^2} = \sqrt{86}$$

$$\therefore \vec{u} = \frac{\langle -1, -7, 6 \rangle}{\sqrt{86}} = \boxed{\left\langle \frac{-1}{\sqrt{86}}, \frac{-7}{\sqrt{86}}, \frac{6}{\sqrt{86}} \right\rangle}$$

[5]

(b) The three points in part (a) correspond to the terminal points of position vectors (that is, vectors with initial point at $(0,0,0)$). Determine the volume of the parallelepiped defined by the vectors.

Let $\vec{a} = \langle 0, 0, -3 \rangle$, $\vec{b} = \langle 4, 2, 0 \rangle$, $\vec{c} = \langle 3, 3, 1 \rangle$

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

$$= \left| \det \begin{bmatrix} 0 & 0 & -3 \\ 4 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix} \right|$$

$$= |0(2-0) - 0(4-0) - 3(12-6)|$$

$$= \boxed{18}$$

[5]